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# When Do State-Dependent Local Projections Work?\*

Sílvia Gonçalves<sup>†</sup>, Ana María Herrera<sup>‡</sup>, Lutz Kilian<sup>§</sup> and Elena Pesavento<sup>±</sup>

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## Abstract

Many empirical studies estimate impulse response functions that depend on the state of the economy. Most of these studies rely on a variant of the local projection (LP) approach to estimate the state-dependent impulse response functions. Despite its widespread application, the asymptotic validity of the LP approach to estimating state-dependent impulse responses has not been established to date. We formally derive this result for a structural state-dependent vector autoregressive process. The model only requires the structural shock of interest to be identified. A sufficient condition for the consistency of the state-dependent LP estimator of the response function is that the first- and second-order conditional moments of the structural shocks are independent of current and future states, given the information available at the time the shock is realized. This rules out models in which the state of the economy is a function of current or future realizations of the outcome variable of interest, as is often the case in applied work. Even when the state is a function of past values of this variable only, consistency may hold only at short horizons.

**JEL codes:** C22, C32, C51

**Keywords:** local projection, state-dependent impulse responses, threshold, identification, nonlinear VAR.

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# 1 Introduction

The recent empirical macroeconomics literature has emphasized the importance of allowing for nonlinearities when estimating the effects of exogenous shocks on macroeconomic variables of interest. A key question in empirical work is how impulse response functions depend on the state of the economy. For example, many studies estimating the government spending multiplier allow for the possibility that this multiplier may be different during recessions and expansions (e.g., Auerbach and Gorodnichenko (2012, 2013a,b), Bachmann and Sims (2012), Owyang, Ramey and Zubairy (2013), Caggiano, Castelnuovo, Colombo and Nodari (2015), Ramey and Zubairy (2018), Alloza (2022), and Ghassibe and Zanetti (2020)). There is also a related literature on the dependence of tax multipliers on the business cycle (e.g., Candelon and Lieb (2013), Alesina, Azzalini, Favero, Giavazzi and Miano (2018), Sims and Wolff (2018), Eskandari (2019), and Demirel (2021)). Similar questions arise in many other contexts including the analysis of monetary policy shocks. For example, Santoro, Petrella, Pfajfar and Gaffeo (2014), Tenreyro and Thwaites (2016), Angrist, Jordà and Kuersteiner (2018), Barnichon and Matthes (2018) and Klepacz (2020) allow the responses to monetary policy shocks to vary as a function of the state of the economy. Other studies allow these responses to vary depending on whether the zero lower bound is binding (e.g., Ramey and Zubairy 2018, Mavroeidis 2021). Yet another example of the estimation of state-dependent responses is the work of Caggiano, Castelnuovo and Groshenny (2014) who examine the dependence of the effects of uncertainty shocks on whether the economy is in recession or expansion.

Most of these studies rely on a variant of the local projection (LP) approach of Jordà (2005, 2009) (see also Dufour and Renault (1998) and Chan and Sakata (2007)) to estimate the state-dependent impulse response functions. One argument for using state-dependent local projections rather than structural nonlinear vector autoregressive (VAR) models is their computational simplicity. Estimating impulse responses in state-dependent VAR models by numerical methods tends to be computationally more challenging than the estimation of state-dependent local projections by the method of least squares.<sup>1</sup> Yet, despite its widespread application, the validity of the LP approach to estimating state-dependent impulse responses has not been established to date.<sup>2</sup>

In this paper, we clarify the conditions under which the state-dependent LP estimator can be

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<sup>1</sup>For example, Ramey (2016, p. 87) stresses that “if one is interested in estimating state dependent models, the ... local projection method is a simple way to estimate such a model and calculate impulse response functions.”

<sup>2</sup>LPs have become an increasingly popular alternative to VAR based estimators of impulse responses. The original LP estimator, as proposed by Jordà (2005, 2009) did not allow for the impulse response function to change depending on the state of the economy. For a review of the rationale underlying standard linear LPs the reader is referred to Plagborg-Møller and Wolf (2021). In this paper we are not concerned with linear approximations to nonlinear processes as in Plagborg-Møller and Wolf (2021), but with approximations that are explicitly state dependent and hence nonlinear.

expected to recover the population impulse responses in multivariate models. Our analysis only requires the structural shock of interest to be identified, allowing the user to remain agnostic about the identification of the remainder of the structural model. As it turns out, the crucial condition for the validity of the LP estimator in this context relates to the information set used to compute the state indicators. If this set only includes exogenous variables determined outside of the model, the state-dependent LP estimator is asymptotically valid and recovers the conditional IRF at any finite horizon. If instead the state indicator is a function of endogenous model variables, the asymptotic validity of the LP estimator depends on whether the state of the economy is a function of current, lagged or future realizations of the endogenous model variables. For example, if the state depends on current values of these variables, the LP estimator asymptotically recovers the impact response, but not necessarily the responses at horizons greater than zero. Basing the state only on lagged values instead allows the LP estimator to consistently estimate impulse responses at longer horizons. The longer the horizon of interest, the more restrictive the lag structure needs to be. In particular, to identify impulse responses up to order  $h_{max}$ , the minimum lag order should be  $h_{max}$ . Put differently, to be able to identify impulse responses at horizon  $h = 0, 1, \dots, h_{max}$ , the state indicator  $H_t$  has to be a function of  $y_{t-h_{max}}, y_{t-h_{max}-1}, \dots$

While these results do not formally establish the inconsistency of the LP estimator when our sufficient conditions are violated, we show by simulation that the LP estimator of the response function tends to be asymptotically biased except for the impact response, when the state of the economy is endogenous. These asymptotic biases may become substantial when cumulating the level responses of the model variables, as required for computing fiscal or monetary multipliers, for example.

State-dependent local projections are extremely popular in macroeconomics because they are easily implemented and because they are believed to be more robust to dynamic model misspecification than numerical estimates of impulse response functions obtained from state-dependent structural VAR models. Our results suggest that researchers need to think carefully about the model specification underlying these local projections. Assessing the validity of the state-dependent LP estimator requires the user to state the underlying structural data generating process.

Our results have important implications for applied work. Of particular concern is that in many macroeconomic applications one would expect exogenous shocks to affect not only the future realizations of the model variables, but also the future state of the economy, rendering the state of the economy endogenous with respect to the model variables. The implicit assumption in many empirical studies is that the state of the economy is exogenous with respect to the model variables. This assumption often is empirically implausible. For example, in models that include log real GDP and

express the state of the economy as a function of the unemployment rate, as in Ramey and Zubairy (2018), the unemployment rate changes systematically with the current log-level of real GDP. This renders the state of the economy endogenous with respect to the model variables.

The exogeneity assumption is also implausible when including log real GDP among the endogenous model variables, while measuring expansions and recessions of the economy based on the deviations of log real GDP from a two-sided HP filter trend, which makes the state of the economy dependent on past, current and future realizations of the endogenous model variables (e.g., Auerbach and Gorodnichenko 2013a). Similarly, exogenously imposing NBER business cycle dates, as in Ramey and Zubairy (2018), is inconsistent with the state of the business cycle depending on the response of the model variables to an exogenous shock, since these model variables are correlated with the data underlying the NBER business cycle definition. Defining the state of the business cycle based on one-sided moving average filters, say, by defining a recession as two successive quarters of negative real GDP growth or by defining the business cycle based on the deviation from a one-sided HP filter trend, as in Alloza (2022), does not materially change this result.

The remainder of the paper is organized as follows. In Section 2, we describe the state-dependent structural model of interest in this paper and define the conditional impulse response function. As is customary in applied work, this response function conditions on the state of the economy in the most recent period, but not on the state of the economy in the current period or in future periods. In Section 3, we define the state-dependent LP estimator of this response function and provide sufficient conditions for its consistency. Section 4 explains why this estimator is not expected to be asymptotically valid in general, when the state of the economy is endogenous with respect to the model variables. We show by simulation that in this case, the state-dependent LP estimator tends to be asymptotically valid in the impact period, but not at longer horizons. We also quantify the asymptotic bias of the LP estimator of the response function for several DGPs. The concluding remarks are in section 5.

## 2 Framework

### 2.1 The model

Let  $z_t \equiv (x_t, y_t)'$  denote an  $n \times 1$  vector of strictly stationary time series, where  $y_t$  is  $k \times 1$  with  $k = n - 1$ . We consider a structural state-dependent VAR process of the form

$$C_{t-1}z_t = \mu_{t-1} + B_{t-1}(L)z_{t-1} + \varepsilon_t, \quad (1)$$

where  $\varepsilon_t = (\varepsilon_{1t}, \varepsilon'_{2t})'$  defines the vector of mutually independent structural shocks. Let

$$B_{t-1}(L) = B_{1,t-1} + B_{2,t-1}L + \dots + B_{p,t-1}L^{p-1},$$

where  $p$  denotes the polynomial lag order. For later convenience, we partition  $B_{t-1}(L)$  conformably with  $z_t$  as

$$B_{t-1}(L) = \begin{pmatrix} B_{11,t-1}(L) & B_{12,t-1}(L) \\ B_{21,t-1}(L) & B_{22,t-1}(L) \end{pmatrix}$$

where  $\mathcal{A}_{ij}$  denotes the  $(i, j)$  block of any partitioned matrix  $\mathcal{A}$ .

All model coefficients evolve over time depending on the state of the economy. In the simplest case, there are only two states (such as a recession and an expansion). Let

$$\begin{aligned} \mu_{t-1} &= \mu_E H_{t-1} + \mu_R (1 - H_{t-1}), \\ C_{t-1} &= C_E H_{t-1} + C_R (1 - H_{t-1}), \text{ and} \\ B_{j,t-1} &= B_{jE} H_{t-1} + B_{jR} (1 - H_{t-1}) \text{ for } j = 1, \dots, p, \end{aligned}$$

where  $H_{t-1}$  is a binary stationary time series that takes the value 1 if the economy is in expansion and 0 otherwise. Unlike in Markov switching models,  $H_{t-1}$  is observed.<sup>3</sup>

We are interested in the response of  $\{y_{t+h} : h = 0, 1, \dots, h_{\max}\}$  to a one-time shock in  $\varepsilon_{1t}$ , conditionally on observing  $H_{t-1} = 1$  or  $H_{t-1} = 0$ . Here,  $h_{\max}$  denotes the largest horizon of the impulse response function of interest. To identify this conditional impulse response, we need to impose further restrictions on the model coefficients. In particular, we postulate that

$$C_{t-1} = \begin{pmatrix} 1 & 0 \\ -C_{21,t-1} & C_{22,t-1} \end{pmatrix}, \quad (2)$$

where  $C_{21,t-1}$  is  $k \times 1$  and  $C_{22,t-1}$  is a  $k \times k$  non-singular matrix whose diagonal elements are 1 by a standard normalization condition. Under these assumptions,  $x_t$  is predetermined with respect to  $y_t$ . Note that we do not restrict  $C_{22,t-1}$  to be lower triangular, which allows  $C_{t-1}$  to be block recursive. Hence, the model is only partially identified in that only the responses to  $\varepsilon_{1t}$  are identified.

Model (1) covers several empirically relevant strategies for identifying the structural shock  $\varepsilon_{1t}$  (and the corresponding conditional IRF of  $y_{t+h}$  with respect to  $\varepsilon_{1t}$ ). One is the narrative approach to identification which uses information extraneous to the model to measure  $\varepsilon_{1t}$ , in which case  $x_t = \varepsilon_{1t}$ . For instance, it is popular to use the narrative approach when identifying monetary policy shocks (e.g., Romer and Romer (1989), Tenreyro and Thwaites (2016)) and fiscal policy shocks (e.g., Ramey and Shapiro (1998), Ramey (2011), Ramey (2016)). Alternatively, the structural shock  $\varepsilon_{1t}$  may be identified via an exclusion restriction that precludes  $x_t$  from responding contemporaneously to the structural shocks in the remaining variables of the system. In this case, the structural shock  $\varepsilon_{1t}$  is identified within the nonlinear SVAR model. One example is Blanchard and Perotti (2002), whose

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<sup>3</sup>Following the applied literature (e.g., Auerbach and Gorodnichenko (2012, 2013a,b), Alloza (2009)), we index the parameters for the system at time  $t$  with  $t - 1$ . This reflects the fact that these parameters depend on  $H_{t-1}$ .

exogenous shocks to government spending ( $\varepsilon_{1t}$ ) are identified by assuming that government spending ( $x_t$ ) does not react within the period to shocks to output and tax revenues ( $y_t$ ). Finally, note that our general model also accommodates the special case of  $x_t$  being an exogenous serially correlated observable variable, as in Alloza, Gonzalo and Sanz (2021).

The structural model for  $z_t$  can be rewritten as

$$\begin{cases} x_t = \mu_{1,t-1} + B_{11,t-1}(L)x_{t-1} + B_{12,t-1}(L)y_{t-1} + \varepsilon_{1t} \\ C_{22,t-1}y_t = \mu_{2,t-1} + C_{21,t-1}x_t + B_{21,t-1}(L)x_{t-1} + B_{22,t-1}(L)y_{t-1} + \varepsilon_{2t}. \end{cases} \quad (3)$$

Without further restrictions (such as postulating that  $C_{22,t-1}$  is lower triangular), the parameters in the equations for  $y_t$  are not identified. However, the fact that  $\varepsilon_{1t}$  is identified suffices to identify the conditional response function of  $y_t$  to a one-time shock in  $\varepsilon_{1t}$ .

We impose the following standard martingale difference sequence (m.d.s.) assumption on the structural errors  $\varepsilon_t$ .

**Assumption 1** *Let  $\mathcal{F}^{t-1} = \sigma(z_{t-1}, H_{t-1}, z_{t-2}, H_{t-2}, \dots)$ . Then  $\varepsilon_t | \mathcal{F}^{t-1} \sim (0, \Sigma)$ , where  $\Sigma$  is a diagonal matrix with diagonal elements given by  $\sigma_i^2$  for  $i = 1, \dots, n$ .*

Assumption 1 stipulates that the structural errors  $\varepsilon_t$  are a martingale difference sequence (m.d.s.) with respect to  $\mathcal{F}^{t-1}$ , the information set generated by the past realizations of  $z_t$  and  $H_t$ . This assumption is standard and implies that  $\varepsilon_t$  is serially uncorrelated. Assumption 1 rules out conditional heteroskedasticity in  $\varepsilon_t$  by assuming that  $\Sigma$  is constant. This assumption turns out to be important for establishing the consistency of state-dependent local projections, as we will explain later. Finally, the assumption that  $\Sigma$  is diagonal implies that the structural errors are mutually uncorrelated, as is standard in the structural VAR literature.

## 2.2 Conditional impulse response function

Consistent with the empirical literature, our goal is to define the causal effect on  $y_{t+h}$  of a one-time shock in  $\varepsilon_{1t}$ , conditionally on  $H_{t-1}$ , the state of the economy at time  $t-1$ . The fact that our model is state dependent is reflected in our definition of the conditional IRF. A common approach in the literature on nonlinear impulse response functions (e.g., Gallant, Rossi and Tauchen (1993), Koop, Pesaran and Potter (1996), Potter (2000), Gouriéroux and Jasiak (2005, 2022), Kilian and Vigfusson (2011), Gonçalves et al. (2021)) is to compare, all else equal, two sample paths for the outcome variables of interest, one where  $\varepsilon_{1t}$  is subject to a one-time shock at time  $t$  and another one where no such shock is present. In a state-dependent model such as ours, this would require fixing  $\varepsilon_{2t}$  and  $H_t$  across the two sample paths. This thought experiment is not realistic when  $\varepsilon_t$  is correlated with

current and future values of  $H_t$  because it ignores the possibility that a shock in  $\varepsilon_{1t}$  may change the states of the economy on impact and in the future.

Hence, we define the conditional IRF more generally as follows. We denote by  $\{y_{t+h}\}$  the baseline path that corresponds to the observed data. This is implied by the sequence of structural disturbances and state indicators

$$\mathcal{E} \cup \mathcal{H} = \{\dots, \varepsilon_{1t-1}, \varepsilon_{1t}, \varepsilon_{1t+1}, \dots, \varepsilon_{2t-1}, \varepsilon_{2t}, \varepsilon_{2t+1}, \dots\} \cup \{\dots, H_{t-1}, H_t, H_{t+1}, \dots\}.$$

The other sample path is  $\{y_{t+h}^*\}$ , which is the path implied by an alternative sequence of shocks and state indicators given by

$$\mathcal{E}^* \cup \mathcal{H}^* = \{\dots, \varepsilon_{1t-1}, \varepsilon_{1t}^*, \varepsilon_{1t+1}, \dots, \varepsilon_{2t-1}, \varepsilon_{2t}, \varepsilon_{2t+1}, \dots\} \cup \{\dots, H_{t-1}, H_t^*, H_{t+1}^*, \dots\}.$$

With this choice of structural shocks and state indicators, the two sample paths are identical prior to shock in  $\varepsilon_{1t}$ . At time  $t$ , the shock hits  $\varepsilon_{1t}$ , yielding  $\varepsilon_{1t}^* = \varepsilon_{1t} + \delta$ , where  $\delta$  is the size of the shock. All other shocks are kept the same. This choice of perturbation is consistent with the assumption that structural shocks are mutually uncorrelated. However, to accommodate the possibility that a shock to  $\varepsilon_{1t}$  may change current and future states, we allow for  $H_s^* \neq H_s$  for  $s \geq t$  when defining  $\mathcal{H}^*$ . If the states are exogenous (in a sense made precise in the next section), we can set  $H_s^* = H_s$  for all  $s$ , in which case  $\mathcal{H}^* = \mathcal{H}$ .

**Remark 1** *One alternative approach to defining the baseline and counterfactual sample paths is to introduce a formal model for  $H_t$  as a function of variables in  $z_t$ . For instance, we could define  $H_t = 1(y_t > 0)$ , as in Section 4. In this case,  $H_s^* \neq H_s$  for  $s \geq t$  by the model assumption on  $H_t$ , and the counterfactual sample paths can be defined as a function of the structural shocks only. Because the state-dependent LP estimator does not require an explicit model for  $H_t$ , our general definition of the CIRF accounts for this possibility by assuming that  $\mathcal{H}^*$  may differ from  $\mathcal{H}$ .*

Our definition of conditional IRF is given next.

**Definition 1** *The conditional impulse response function of  $y_{t+h}$  to a one-time shock of size  $\delta$  in  $\varepsilon_{1t}$  is given by  $CIRF_h(H_{t-1}) = E[y_{t+h}^* - y_{t+h} | H_{t-1}]$ , for  $h = 0, 1, 2, \dots, h_{\max}$ .*

Note that Definition 1 conditions only on  $H_{t-1}$ , the state of the economy in the period prior to the shock.<sup>4</sup> This shows that the conditional IRF depends on the state of the economy at time  $t - 1$ , but not on the current or future states of the economy. Nor do we condition on the history of states

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<sup>4</sup>The conditional expectation is defined with respect to the distribution of  $\{\varepsilon_s\} \cup \{H_s : s \neq t - 1\} \cup \{H_s^* : s \geq t\}$ , given  $H_{t-1}$ . This expectation is time invariant by the stationarity of  $(z_t, H_t)$ , which we assume throughout.



prior to  $t-1$ . Rather, we average them out, conditioning only on the previous state. This corresponds to the standard approach in estimating state-dependent responses in applied macroeconomics, where interest centers on the question of how the IRF differs, depending on whether the economy was in expansion or recession prior to the shock.

**Remark 2** *Although we focus on Definition 1 in this paper, it is worth noting that other definitions of impulse response functions may be considered in nonlinear settings such as ours. One possibility is the unconditional IRF, as in Gonçalves et al. (2021), defined as  $IRF_h \equiv E(y_{t+h}^* - y_{t+h})$ . Another possibility is an IRF that conditions on the information set  $\mathcal{G}^{t-1}$  available at time  $t-1$ . One example is to let  $\mathcal{G}^{t-1} = \mathcal{F}^{t-1}$ , the history at time  $t-1$ , including the value of  $H_{t-1}$ . This includes Definition 1 as a special case with  $\mathcal{G}^{t-1} = \{H_{t-1}\}$ . Conditioning only on  $H_{t-1}$  is common in applied macroeconomics. This convention allows researchers to report only two types of IRFs, depending on whether the economy was in an expansion or in a recession prior to the shock. As we show in the next section, local projections that involve interactions of  $H_{t-1}$  and  $x_t$  recover IRFs conditional on  $\mathcal{G}^{t-1} = \{H_{t-1}\}$ .*

Although the counterfactual  $y_{t+h}^*$  is not observed, it may be recovered from the structural model given  $\mathcal{E}^*$  and  $\mathcal{H}^*$ . The values of  $y_{t+h}^*$  obtained from solving the model given these sequences is related to the notion of potential outcomes, as defined by Angrist and Kuersteiner (2011) and Angrist, Jordá and Kuersteiner (2016). Further discussion of potential outcomes for time series processes can be found in White (2016) and Rambachan and Shephard (2021).

### 3 What happens when $H_t$ is exogenous?

#### 3.1 Expression for CIRF

In this section, we present an expression for  $CIRF_h(H_{t-1})$  for the state-dependent structural model given in (3). For expositional purposes, we set  $\delta = 1$ . We focus on a counterfactual that treats  $H_t$  as exogenous with respect to  $\varepsilon_t$  such that  $\mathcal{H}^* = \mathcal{H}$  in Definition 1. To describe the population IRF, we evaluate the difference between  $y_{t+h}^*$  and  $y_{t+h}$ . Since  $C_{t-1}$  satisfies the identification condition (2), the inverse matrix of  $C_{t-1}$  exists and is given by

$$C_{t-1}^{-1} = \begin{pmatrix} 1 & 0 \\ C_{22,t-1}^{-1}C_{21,t-1} & C_{22,t-1}^{-1} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ C_{t-1}^{21} & C_{t-1}^{22} \end{pmatrix},$$

where for any matrix  $\mathcal{A}$ , we let  $\mathcal{A}^{ij}$  denote the block  $(i, j)$  of  $\mathcal{A}^{-1}$ .

Pre-multiplying (1) by  $C_{t-1}^{-1}$  yields

$$z_t = C_{t-1}^{-1}\mu_{t-1} + C_{t-1}^{-1}B_{t-1}(L)z_{t-1} + C_{t-1}^{-1}\varepsilon_t,$$

which we rewrite as

$$z_t = b_{t-1} + A_{t-1}(L)z_{t-1} + \eta_t, \quad (4)$$

where  $\eta_t \equiv C_{t-1}^{-1}\varepsilon_t$ ,  $b_{t-1} \equiv C_{t-1}^{-1}\mu_{t-1}$ , and

$$A_{t-1}(L) \equiv C_{t-1}^{-1}B_{t-1}(L) = A_{1,t-1} + A_{2,t-1}L + \dots + A_{p,t-1}L^{p-1},$$

with  $A_{j,t-1} \equiv C_{t-1}^{-1}B_{j,t-1}$ .

The value of  $y_{t+h}$  and  $y_{t+h}^*$  can be obtained from the companion-form representation of the reduced-form model (4). Let

$$Z_t = \begin{pmatrix} z_t' \\ z_{t-1}' \\ \dots \\ z_{t-p+1}' \end{pmatrix}'_{np \times 1}, \quad \xi_t = \begin{pmatrix} \eta_t' \\ 0' \end{pmatrix}'_{np \times 1}, \quad a_{t-1} = \begin{pmatrix} b_{t-1}' \\ 0' \end{pmatrix}'_{np \times 1},$$

and

$$A_{t-1} = \begin{pmatrix} A_{1,t-1} & A_{2,t-1} & \dots & A_{p-1,t-1} & A_{p,t-1} \\ I_n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{pmatrix}.$$

We can rewrite (4) as

$$Z_t = a_{t-1} + A_{t-1}Z_{t-1} + \xi_t. \quad (5)$$

To obtain  $y_t$  from  $Z_t$ , let

$$\mathbb{S}_k = \begin{pmatrix} 0_{k \times 1} & I_k & 0_{k \times n(p-1)} \end{pmatrix}$$

denote a  $k \times np$  selection matrix (with  $k = n - 1$  equal to the number of variables in  $y_t$ ) which selects the subvector  $y_t$  from the vector  $Z_t$ . With this notation,

$$y_t = \mathbb{S}_k Z_t,$$

and, more generally, for any  $h$ ,

$$y_{t+h} = \mathbb{S}_k Z_{t+h}.$$

Note that for  $k = 1$  (i.e., for a bivariate system with  $n = 2$ ),  $\mathbb{S}_k = e'_{2,2p}$ , where  $e_{2,2p} = (0, 1, 0')$  is a  $2p \times 1$  vector whose only non-zero element is equal to 1 and occurs in position 2. More generally, we let  $e_{j,m}$  denote a  $m \times 1$  vector with 1 in position  $j$  and 0 elsewhere.

Next, we use the companion form (5) to obtain the difference  $y_{t+h}^* - y_{t+h}$  for different values of  $h$ . Starting with  $h = 0$ , and noting that the two sample paths coincide up to time  $t - 1$ , we have that

$$Z_t = a_{t-1} + A_{t-1}Z_{t-1} + \xi_t \quad \text{and} \quad Z_t^* = a_{t-1} + A_{t-1}Z_{t-1} + \xi_t^*.$$

Hence,

$$Z_t^* - Z_t = \xi_t^* - \xi_t = \begin{pmatrix} \eta_t^* - \eta_t \\ 0_{n(p-1) \times 1} \end{pmatrix},$$

where  $\eta_t^* - \eta_t = C_{t-1}^{-1}(\varepsilon_t^* - \varepsilon_t)$ . Since we only perturb the first element of  $\varepsilon_t$ , the following decomposition of  $\eta_t$  is useful:

$$\eta_t \equiv C_{t-1}^{-1}\varepsilon_t = \begin{pmatrix} 1 \\ C_{t-1}^{21} \end{pmatrix} \varepsilon_{1t} + \begin{pmatrix} 0 \\ C_{t-1}^{22} \end{pmatrix} \varepsilon_{2t} \equiv C_{t-1}^{-1}e_{1n}\varepsilon_{1t} + C_{t-1}^{-1}I_{2:n}\varepsilon_{2t},$$

where  $e_{1n} \equiv (1, 0')'$  is  $n \times 1$  and  $I_{2:n}$  is  $k \times n$  and is equal to the  $n \times n$  identity matrix with its first column removed:

$$I_{2:n} = \begin{pmatrix} e_{2n} & \cdots & e_{nn} \end{pmatrix}.$$

With this notation,

$$\eta_t^* - \eta_t = C_{t-1}^{-1}e_{1n} \underbrace{(\varepsilon_{1t}^* - \varepsilon_{1t})}_{=1} + C_{t-1}^{-1}I_{2:n} \underbrace{(\varepsilon_{2t}^* - \varepsilon_{2t})}_{=0} = C_{t-1}^{-1}e_{1n},$$

given our definition of  $\mathcal{E}$  and  $\mathcal{E}^*$ . It follows that

$$Z_t^* - Z_t = \begin{pmatrix} \eta_t^* - \eta_t \\ 0_{n(p-1) \times 1} \end{pmatrix} = \begin{pmatrix} C_{t-1}^{-1}e_{1n} \\ 0_{n(p-1) \times 1} \end{pmatrix} = e_{1p} \otimes C_{t-1}^{-1}e_{1n},$$

and, consequently,

$$y_t^* - y_t = \mathbb{S}_k(Z_t^* - Z_t) = \mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}e_{1n}).$$

The conditional response at  $h = 0$  is given by

$$CIRF_0(H_{t-1}) = E(y_t^* - y_t | H_{t-1}) = \mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}e_{1n}), \quad (6)$$

since  $C_{t-1}^{-1}$  is known conditionally on  $H_{t-1}$ . In particular, the individual impact responses of each variable in  $y_t$  can be obtained as

$$CIRF_{0,j}(H_{t-1}) = E(y_{jt}^* - y_{jt} | H_{t-1}) = e'_{j,np}(e_{1p} \otimes C_{t-1}^{-1}e_{1n}),$$

for  $j = 2, \dots, n$ . When  $n = 2$  (bivariate system),  $k = 1$ , implying that  $\mathbb{S}_k = e'_{2,2p}$ .

The expression (6) shows that the conditional impact response can take on two different values, depending on whether  $H_{t-1} = 1$  or  $H_{t-1} = 0$ ,

$$CIRF_0(H_{t-1}) = \begin{cases} \mathbb{S}_k(e_{1p} \otimes C_E^{-1}e_{1n}), & \text{if } H_{t-1} = 1 \\ \mathbb{S}_k(e_{1p} \otimes C_R^{-1}e_{1n}), & \text{if } H_{t-1} = 0, \end{cases}$$

since  $C_{t-1} \equiv C_E H_{t-1} + C_R(1 - H_{t-1})$ . It also shows that only the first column of  $C_{t-1}^{-1}$  (i.e.,  $C_{t-1}^{-1}e_{1n}$ ) matters for the identification of the conditional impact response.

For  $h = 1$ , we use the companion form to evaluate first  $Z_{t+1}^* - Z_{t+1}$  and then  $y_{t+1}^* - y_{t+1}$ , as follows. In particular,

$$Z_{t+1} = a_t + A_t Z_t + \xi_{t+1},$$

where  $\xi_{t+1} = (\eta'_{t+1}, 0')' = ((C_t^{-1}\varepsilon_{t+1})', 0')'$ , and where  $a_t$ ,  $A_t$  and  $C_t$  depend on  $H_t$ . Similarly,

$$Z_{t+1}^* = a_t^* + A_t^* Z_t^* + \xi_{t+1}^*,$$

where  $\xi_{t+1}^* = (\eta'^*_{t+1}, 0')' = ((C_t^{*-1}\varepsilon^*_{t+1})', 0')'$  and  $a_t^*$ ,  $A_t^*$  and  $C_t^*$  depend on  $H_t^*$ . Given our choice of structural shocks,  $\varepsilon^*_{t+1} = \varepsilon_{t+1}$ . Moreover, the assumption that  $H_t^* = H_t$  implies  $a_t^* = a_t$ ,  $A_t^* = A_t$  and  $C_t^* = C_t$ . It follows that

$$Z_{t+1}^* - Z_{t+1} = A_t (Z_t^* - Z_t) = A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}),$$

given that  $Z_t^* - Z_t = e_{1p} \otimes C_{t-1}^{-1} e_{1n}$ . Thus, we have that

$$y_{t+1}^* - y_{t+1} = \mathbb{S}_k (Z_{t+1}^* - Z_{t+1}) = \mathbb{S}_k A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}),$$

implying that

$$CIRF_1(H_{t-1}) = \mathbb{S}_k E(A_t | H_{t-1}) (e_{1p} \otimes C_{t-1}^{-1} e_{1n}).$$

This expression generalizes to other values of  $h$  as follows.

**Proposition 3.1** *Let  $\mathcal{E}$ ,  $\mathcal{H}$ ,  $\mathcal{E}^*$  and  $\mathcal{H}^*$  be as defined in Section 2.2. If  $\mathcal{H}^* = \mathcal{H}$ , the impulse response of  $y_t$  to a one-time shock in  $\varepsilon_{1t}$  of size  $\delta = 1$ , conditional on  $H_{t-1}$ , is*

$$CIRF_0(H_{t-1}) \equiv E(y_t^* - y_t | H_{t-1}) = \mathbb{S}_k (e_{1p} \otimes C_{t-1}^{-1} e_{1n}),$$

and for any  $h \geq 1$ ,

$$CIRF_h(H_{t-1}) \equiv E(y_{t+h}^* - y_{t+h} | H_{t-1}) = \mathbb{S}_k E(A_{t+h-1} A_{t+h-2} \dots A_t | H_{t-1}) (e_{1p} \otimes C_{t-1}^{-1} e_{1n}).$$

To identify  $CIRF_h(H_{t-1})$ , we need to identify the first column of  $C_{t-1}^{-1}$ ,  $C_{t-1}^{-1} e_{1n}$ , as well as the coefficients that enter the matrices  $A_{t+h-1}$  through  $A_t$ . Given that these matrices are linear in the state indicators, identification can be achieved from the reduced-form model (4), where  $\varepsilon_{1t}$  is identified from the first equation in the structural model (1) given the identification condition (2). Even when the model is fully identified, evaluating  $E(A_{t+h-1} A_{t+h-2} \dots A_t | H_{t-1})$  is challenging and requires knowledge of the conditional density of  $H_{t+h-1}, \dots, H_t$ , given  $H_{t-1}$ .<sup>5</sup> Local projections are a much simpler alternative and do not require imposing a model assumption on  $H_t$ . In the next section, we provide a set of sufficient conditions under which local projections are consistent.

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<sup>5</sup>Kole and van Dijk (2021) provide closed-form expressions for the first- and second-order moments of a Markov switching SVAR model under the assumption that  $H_t$  is a first-order Markov process. Although they also provide formulas for nonlinear CIRFs, their definition relies on a different counterfactual than ours.

### 3.2 Local projections

A state-dependent LP regression is a direct regression of  $y_{t+h}$  onto a constant,  $x_t$  and  $Z_{t-1}$ , each interacted with  $H_{t-1}$  and  $1-H_{t-1}$ . The slope coefficients associated with  $x_t H_{t-1}$  are usually interpreted as the CIRF of  $y_{t+h}$ , conditionally on  $H_{t-1} = 1$ , whereas the slope coefficients associated with  $x_t(1-H_{t-1})$  are interpreted as the CIRF of  $y_{t+h}$  when we condition on  $H_{t-1} = 0$ . The goal of this section is to provide a set of regularity conditions under which this interpretation is asymptotically valid.

Let  $W_{t-1} \equiv (1, Z'_{t-1})'$  denote an  $(np+1) \times 1$  vector of control variables which include a constant and  $p$  lags of  $z_t$ . A state-dependent LP for identifying the causal effect on  $y_{t+h}$  of a one-time shock in  $\varepsilon_{1t}$  of size  $\delta = 1$  can be written as

$$y_{t+h} = b_{E,h}x_t H_{t-1} + \Pi_{E,h}W_{t-1}H_{t-1} + b_{R,h}x_t(1-H_{t-1}) + \Pi_{R,h}W_{t-1}(1-H_{t-1}) + v_{t+h}, \quad (7)$$

where the  $k \times 1$  vectors  $b_{E,h}$  and  $b_{R,h}$  contain the main parameters of interest. In particular,  $b_{E,h}$  is interpreted as the CIRF of  $y_{t+h}$  when  $H_{t-1} = 1$ , whereas  $b_{R,h}$  contains the CIRF of  $y_{t+h}$ , conditionally on  $H_{t-1} = 0$ . The matrices  $\Pi_{E,h}$  and  $\Pi_{R,h}$  are of size  $k \times (np+1)$ ; each row contains the constant and the slope coefficients associated with  $Z_{t-1}$  for the LP regression of each variable in  $y_{t+h}$ . The LP regression for variable  $y_{j,t+h}$  is

$$y_{j,t+h} = b_{E,j,h}x_t H_{t-1} + \pi'_{E,j,h}W_{t-1}H_{t-1} + b_{R,j,h}x_t(1-H_{t-1}) + \pi'_{R,j,h}W_{t-1}(1-H_{t-1}) + v_{j,t+h}, \quad (8)$$

where  $j = 2, \dots, n$ . The scalar coefficients  $b_{E,j,h}$  and  $b_{R,j,h}$  are the  $(j-1)^{th}$  elements of  $b_{E,h}$  and  $b_{R,h}$ , respectively. Similarly,  $\pi'_{E,j,h}$  and  $\pi'_{R,j,h}$  are the corresponding rows of  $\Pi_{E,h}$  and  $\Pi_{R,h}$ .

Since  $H_t$  is observed, the coefficients in the multivariate state-dependent LP regression (7) can be obtained by running a multivariate LS regression of  $y_{t+h}$  onto  $x_t H_{t-1}$ ,  $W_{t-1}H_{t-1}$ ,  $x_t(1-H_{t-1})$  and  $W_{t-1}(1-H_{t-1})$ . Note that this is equivalent to running a regression of  $y_{j,t+h}$  onto  $x_t H_{t-1}$ ,  $W_{t-1}H_{t-1}$ ,  $x_t(1-H_{t-1})$  and  $W_{t-1}(1-H_{t-1})$ , for each  $j = 2, \dots, n$ . Put differently, the multivariate LS regression (7) is equivalent to the  $k$  univariate OLS regressions (8), equation-by-equation.

Let  $\hat{b}_{E,h}$  and  $\hat{b}_{R,h}$  denote the LS estimators of  $b_{E,h}$  and  $b_{R,h}$  in (7) based on a sample of size  $T$  given by  $\{y_{t+h}, x_t, Z_{t-1}, H_{t-1} : t = 1, \dots, T\}$ . We can estimate each of these vectors separately, by restricting the sample to  $H_{t-1} = 1$  and  $H_{t-1} = 0$ , respectively. For instance,  $\hat{b}_{E,h}$  can be obtained from a regression of  $y_{t+h}$  on  $x_t H_{t-1}$  and  $W_{t-1}H_{t-1}$  (omitting  $x_t(1-H_{t-1})$  and  $W_{t-1}(1-H_{t-1})$  in the regression). This follows because  $H_{t-1}(1-H_{t-1}) = 0$  for all  $t$ . Similarly, we can obtain  $\hat{b}_{R,h}$  from a regression of  $y_{t+h}$  on  $x_t(1-H_{t-1})$  and  $W_{t-1}(1-H_{t-1})$  (omitting  $x_t H_{t-1}$  and  $W_{t-1}H_{t-1}$  in this regression).

As it turns out, Assumption 1 suffices to show the consistency of  $\hat{b}_{E,h}$  and  $\hat{b}_{R,h}$  when  $h = 0$ . To identify the CIRF at horizons  $h = 1, \dots, h_{\max}$  we add the following assumption.

**Assumption 2** Let  $h_{\max} \geq 1$  denote the maximum horizon of the response function of interest. Then, for  $h = 1, \dots, h_{\max}$ ,

$$(a) \quad E(\varepsilon_t | H_{t+h-1}, \dots, H_t, \mathcal{F}^{t-1}) = E(\varepsilon_t | \mathcal{F}^{t-1}).$$

$$(b) \quad E(\varepsilon_t \varepsilon_t' | H_{t+h-1}, \dots, H_t, \mathcal{F}^{t-1}) = E(\varepsilon_t \varepsilon_t' | \mathcal{F}^{t-1}).$$

Assumption 2 characterizes the relationship between the structural shocks  $\{\varepsilon_t\}$  and the state indicators  $\{H_t\}$ . This condition is crucial for proving the validity of state-dependent local projections. A sufficient condition for Assumption 2 is to assume that  $\{H_t\}$  is fully independent of  $\{\varepsilon_s\}$ . This assumption is satisfied if we construct  $H_t$  on the basis of variables not contained in  $z_t$  that are independent of the structural errors  $\varepsilon_t$ . Assumption 2 is a milder assumption than full independence between  $\varepsilon_t$  and  $H_t$ . It only requires the conditional first two moments of  $\varepsilon_t$  to be independent of  $\{H_t, H_{t+1}, \dots, H_{t+h-1}\}$ , conditionally on  $\mathcal{F}^{t-1}$ , where  $h \leq h_{\max}$ . This allows for the possibility that  $H_t$  is obtained as a function of past values of  $z_t$ . How many lags of  $z_t$  can be included in  $H_t$  depends on the value of  $h_{\max}$ . For  $h_{\max} = 1$ ,  $H_t$  can depend on  $z_{t-1}$  (and previous lags of  $z_{t-1}$ ), but for  $h_{\max} = 2$ ,  $H_t$  must not depend on  $z_{t-1}$  (although it can depend on  $z_{t-2}$  or further lags of  $z_{t-2}$ ). As  $h_{\max}$  increases, the set of lags used to construct  $H_t$  shrinks. In the limit, if we are interested in the entire impulse response function,  $H_t$  cannot be chosen as a function of  $\{z_t\}$ . We will further illustrate the content of Assumption 2 in the next section when we specialize  $H_t$  to be a deterministic function of  $z_t$ .

Under Assumptions 1 and 2, we can prove the following result.

**Proposition 3.2** Under Assumptions 1 and 2, as  $T \rightarrow \infty$ , for any  $h = 0, 1, \dots, h_{\max}$ ,

$$\hat{b}_{E,h} \xrightarrow{p} CIRF_h(H_{t-1} = 1) \quad \text{and} \quad \hat{b}_{R,h} \xrightarrow{p} CIRF_h(H_{t-1} = 0),$$

where  $CIRF_h(H_{t-1} = i) = \mathbb{S}_k E(A_{t+h-1} A_{t+h-2} \dots A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}) | H_{t-1} = i)$  for  $i = 1, 0$ .

The proof of Proposition 3.2 is in the Appendix. Proposition 3.2 shows that the LP regression (7) identifies the conditional IRF defined in Proposition 3.1. The latter corresponds to the CIRF of  $y_{t+h}$  derived under the counterfactual experiment that sets  $\mathcal{H}^* = \mathcal{H}$ . In other words, we assume that the shock of  $\varepsilon_{1t}$  does not change the state of the economy on impact or in the future. This is consistent with Assumption 2, which imposes moment-independence conditions on  $\varepsilon_t$  and  $H_{t+h-1}, \dots, H_t$ , conditionally on  $\mathcal{F}^{t-1}$ .

The model equation for  $y_{t+h}$  implied by the structural model may be used to heuristically understand why the state-dependent LP works without Assumption 2 when  $h = 0$ , but not otherwise. More specifically, consider a simplified bivariate version of model (1), where  $x_t = \varepsilon_{1t}$  and

$y_t = \beta_{t-1}x_t + \gamma_{t-1}y_{t-1} + \varepsilon_{2t}$ . Consider first  $h = 0$ . Then, if we condition on  $H_{t-1} = 1$ ,

$$y_t = \beta_E x_t + \underbrace{\gamma_E y_{t-1} + \varepsilon_{2t}}_{=v_t},$$

where  $\beta_E$  is the coefficient associated with  $\varepsilon_{1t}$ , is the conditional IRF when  $H_{t-1} = 1$ . To understand why the state-dependent LP estimator  $\hat{b}_{E,0}$  recovers  $\beta_E$  without further assumptions other than Assumption 1 (and the assumed stationarity and ergodicity of the data), note that the probability limit of  $\hat{b}_{E,h}$  is equal to  $b_{E,h} = \frac{E(x_t y_{t+h} | H_{t-1}=1)}{E(x_t^2 | H_{t-1}=1)}$ , the population OLS coefficient associated with  $x_t = \varepsilon_{1t}$  in a linear regression of  $y_{t+h}$  on  $x_t$  which conditions on  $H_{t-1} = 1$ . For  $h = 0$ , this coefficient is  $\beta_E$  provided the error term  $v_t \equiv \gamma_E y_{t-1} + \varepsilon_{2t}$  is orthogonal to  $x_t$ , conditionally on  $H_{t-1} = 1$ . This orthogonality condition holds under the m.d.s. assumption on  $\varepsilon_t$  (i.e., Assumption 1) without further restrictions on  $H_t$ .

For  $h = 1$ , conditionally on  $H_{t-1} = 1$ , the model equation for  $y_{t+1}$  now is

$$y_{t+1} = \beta_t x_{t+1} + \gamma_t (\beta_E x_t + \gamma_E y_{t-1} + \varepsilon_{2t}) + \varepsilon_{2t+1} = \gamma_t \beta_E x_t + v_{t+1},$$

where  $v_{t+1} = \gamma_t \gamma_E y_{t-1} + \gamma_t \varepsilon_{2t} + \beta_t x_{t+1} + \varepsilon_{2t+1}$  depends on  $H_t$  through  $\gamma_t$  and  $\beta_t$ . Hence, without further assumptions that restrict the dependence between  $H_t$  and  $\varepsilon_t$ , we cannot conclude that  $v_{t+1}$  is orthogonal to  $x_t$ , conditionally on  $H_{t-1} = 1$ . Assumption 2(a) together with Assumption 1 ensures that this is true, i.e., that  $E(x_t v_{t+1} | H_{t-1} = 1) = 0$ . We obtain  $E(x_t y_{t+1} | H_{t-1} = 1) = E(\gamma_t x_t^2 | H_{t-1} = 1) \beta_E$ . To conclude that the state-dependent LP estimand  $b_{E,1} = \frac{E(x_t y_{t+1} | H_{t-1}=1)}{E(x_t^2 | H_{t-1}=1)}$  equals  $E(\gamma_t | H_{t-1} = 1) \beta_E$ , we further impose Assumption 2(b). In particular, by the LIE, we can write  $E(\gamma_t x_t^2 | H_{t-1} = 1) = E(\gamma_t E(x_t^2 | H_t, \mathcal{F}^{t-1}) | H_{t-1} = 1)$ . Using Assumption 2(b) and the conditional homoskedasticity assumption on  $\varepsilon_{1t}$ ,  $E(x_t^2 | H_t, \mathcal{F}^{t-1}) = E(x_t^2 | \mathcal{F}^{t-1}) = \sigma_1^2$ . This implies that the LP estimand for  $h = 1$  is equal to  $E(\gamma_t | H_{t-1} = 1) \beta_E$ , the conditional IRF for  $h = 1$  derived in Proposition 3.1. It is worth noting that this result relies not only on the conditional moment independence assumption between  $\varepsilon_t$  and  $H_t$  (Assumption 2(b)), but also on the conditional homoskedasticity assumption on  $\varepsilon_{1t}$ .

For general values of  $h$ , we can write  $y_{t+h}$  as a function of  $x_t$  and an error term that depends on  $H_{t+h-1}, \dots, H_{t-1}$ . Conditionally on  $H_{t-1}$ , this is a state-dependent equation, as it depends on  $H_{t+h-1}, \dots, H_t$ . A linear local projection of  $y_{t+h}$  on  $x_t$  which conditions only on  $H_{t-1}$  recovers the conditional IRF derived in Proposition 3.1 provided the error term is orthogonal to  $\varepsilon_{1t}$ , conditionally on  $H_{t-1}$ . Since this error depends on  $H_{t+h-1}, \dots, H_t$ , we require that  $\varepsilon_{1t}$  be independent of  $H_{t+h-1}, \dots, H_t$ , conditionally on  $H_{t-1}$ . Assumption 2 formalizes this independence condition. Because local projections are least squares estimates, it is natural that only first- and second-order conditional moment independence conditions on  $\varepsilon_t$  are required. As for  $h = 1$ , the conditional ho-

maskedasticity assumption implied by Assumption 1 is also important for deriving the consistency of the state-dependent LP estimator for general values of  $h \geq 1$ .

Note that the asymptotic validity of the state-dependent LP estimator does not depend on the full identification of the structural model parameters. The crucial condition is that the shock of interest  $\varepsilon_{1t}$  is identified. With this condition, and under Assumptions 1 and 2, the LP identifies the correct conditional IRF even though the contemporaneous state-dependent matrix  $C_{t-1}$  is only block recursive. This result is expected. Proposition 3.1 shows that the conditional IRF depends on the conditional expectation of a function of  $A_{t+h-1}, \dots, A_t$  and the first column of  $C_{t-1}^{-1}$ , which can all be identified from the reduced-form model (4).

## 4 What happens when $H_t$ is endogenous?

In this section, we investigate the properties of state-dependent LPs when  $H_t$  does not satisfy Assumption 2. In particular, we consider the case when  $H_t$  depends on current values of the outcome variables  $y_t$ . To simplify the exposition and make the arguments clearer, we consider the special case of a bivariate structural model for  $z_t \equiv (x_t, y_t)'$ , where  $x_t$  is a directly observed shock and  $y_t$  has limited dynamics:

$$\begin{cases} x_t = \varepsilon_{1t}, \\ y_t = \beta_{t-1}x_t + \gamma_{t-1}y_{t-1} + \varepsilon_{2t}. \end{cases} \quad (9)$$

In terms of our previous notation,  $n = 2$ ,  $k = 1$ ,  $C_{22,t-1} = 1$ ,  $C_{21,t-1} = \beta_{t-1}$ ,  $C_{21,t-1}(L) = 0$  and  $B_{22,t-1}(L) = \gamma_{t-1}$ . The state-dependent parameters  $\beta_{t-1}$  and  $\gamma_{t-1}$  depend on  $H_{t-1}$  as before. For instance,  $\beta_{t-1} = \beta_E H_{t-1} + \beta_R(1 - H_{t-1})$ . Crucially, we now endogenize  $H_t$  with respect to the structural shocks  $\varepsilon_t$ . In particular, we let  $H_t = 1(y_t > 0)$ . Given that the structural model sets the time  $t$  coefficients as a function of  $H_{t-1}$ , as is typically assumed in the empirical literature, setting  $H_t = 1(y_t > 0)$  implies that  $\beta_{t-1}$  and  $\gamma_{t-1}$  are a function of  $y_{t-1}$ . A generalization of this scenario is to allow  $H_t$  to depend on current and lagged values of  $y_t$ , as in Alloza's (2019) study of the impact of a fiscal policy shock on output. Alloza sets  $H_t = 1(y_t > 0 \text{ or } y_{t-1} > 0)$ .

Next, we discuss the implications of this choice of  $H_t$  for the validity of the LP estimator. First, we show that the conditional IRF of interest is no longer given by the formula derived in Proposition 3.1. Next, we argue why the LP estimand is not the same as the one derived in Proposition 3.2. Finally, because an analytical characterization of this estimand is infeasible, we numerically illustrate the magnitude of the asymptotic bias of the state-dependent LP estimator in this context.



## 4.1 Conditional IRF when $H_t$ is endogenous

As before, the goal is to obtain the response of  $y_{t+h}$  to a shock of size 1 in  $\varepsilon_{1t}$ . We follow the same approach as in Section 2.2 and compare the value of  $y_{t+h}$  with a counterfactual value  $y_{t+h}^*$  which corresponds to what we would have observed if we had perturbed  $\varepsilon_{1t}$  by 1 without changing any of the other inputs to the system. Note that when  $H_t$  depends on  $y_t$ , the current and future values of  $H_t$  cannot be kept constant across these two sample paths. Thus, a counterfactual experiment that sets  $\mathcal{H}^* = \mathcal{H}$  is not consistent with this choice of  $H_t$ . We need to account for the impact of the shock in  $\varepsilon_{1t}$  on the current and future values of the states of the economy such that  $H_s^* \neq H_s$  for  $s \geq t$ .

Consider  $h = 0$ . Following the same steps as in Section 3.1, we can show that

$$y_t^* - y_t = \beta_{t-1} (x_t^* - x_t) = \beta_{t-1} \equiv \beta_E H_{t-1} + \beta_R (1 - H_{t-1}),$$

since  $x_t^* = x_t + 1$ , and importantly,  $\beta_{t-1}^* = \beta_{t-1}$  and  $\gamma_{t-1}^* = \gamma_{t-1}$ . This follows because  $\beta_{t-1}^*$  and  $\gamma_{t-1}^*$  are defined as  $\beta_{t-1}$  and  $\gamma_{t-1}$ , but depend on  $H_{t-1}^* = 1 (y_{t-1}^* > 0) = H_{t-1} = 1 (y_{t-1} > 0)$  since  $y_{t-1}^* = y_{t-1}$ . This implies that the conditional impact response defined in Proposition 3.1 is

$$CIRF_0(H_{t-1}) = \beta_{t-1} = \begin{cases} \beta_E & \text{if } H_{t-1} = 1 \\ \beta_R & \text{if } H_{t-1} = 0. \end{cases}$$

To see that this expression is a special case of Proposition 3.1, note that when  $k = 1$ ,  $\mathbb{S}_k = (0, 1)$ ,  $e_{1p} = 1$  and  $C_{t-1}^{-1} e_{1n} = (1, \beta_{t-1})'$ .

For  $h = 1$ , an important difference emerges. Now,  $\beta_t^*$  and  $\gamma_t^*$  depend on  $H_t^* = 1 (y_t^* > 0)$ . Since  $y_t^*$  is not equal to  $y_t$ , we cannot set  $\beta_t^* = \beta_t$  and  $\gamma_t^* = \gamma_t$  when defining the counterfactual value of  $y_{t+1}$ . In particular, we now have

$$y_{t+1}^* = \beta_t^* x_{t+1}^* + \gamma_t^* y_t^* + \varepsilon_{2t+1}^* = \beta_t^* x_{t+1} + \gamma_t^* y_t^* + \varepsilon_{2t+1},$$

where the second equality follows because  $\varepsilon_{2t+1}^* = \varepsilon_{2t+1}$ , and  $x_{t+1}^* = \varepsilon_{1t+1}^* = \varepsilon_{1t+1}$ . The difference between  $y_{t+1}^*$  and  $y_{t+1}$  is

$$y_{t+1}^* - y_{t+1} = (\beta_t^* - \beta_t) x_{t+1} + (\gamma_t^* - \gamma_t) y_t^* + \gamma_t (y_t^* - y_t),$$

where  $y_t^* - y_t = \beta_{t-1}$ . The fact that  $H_t$  is a function of  $y_t$  implies that a shock at time  $t$  in  $\varepsilon_{1t}$  has an impact on  $y_t$  and hence an impact on the state-dependent coefficients  $\beta_t^*$  and  $\gamma_t^*$ . This explains the presence of the two extra terms in  $y_{t+1}^* - y_{t+1}$ .

The conditional impulse response at horizon  $h = 1$  is the expectation of this difference, conditionally on  $H_{t-1}$ :

$$\begin{aligned} CIRF_1(H_{t-1}) &= E(y_{t+1}^* - y_{t+1} | H_{t-1}) \\ &= \underbrace{E[(\beta_t^* - \beta_t) x_{t+1} | H_{t-1}] + E[(\gamma_t^* - \gamma_t) y_t^* | H_{t-1}]}_{\text{Indirect effect}} + \underbrace{E(\gamma_t | H_{t-1}) \beta_{t-1}}_{\text{Direct effect}}. \end{aligned}$$

The second term corresponds to the CIRF derived in Proposition 3.1 under the assumption that the counterfactual value of  $H_t^*$  is equal to the observed value  $H_t$ , i.e.,  $\mathcal{H} = \mathcal{H}^*$ . We interpret this as the direct effect as it captures the effect on  $y_{t+1}$  of the shock in  $\varepsilon_{1t}$  assuming that there is no change in the state  $H_t$  (and therefore no change in  $\beta_t$  and  $\gamma_t$ ). The first term accounts for the effect on  $y_{t+1}$  that occurs because  $H_t$  has changed. When  $H_t$  is exogenous, this indirect effect is zero, but not otherwise.

Note that we can use the model equations to express the indirect effect as a function of observables. In particular, it can be shown that<sup>6</sup>

$$\text{Indirect effect} = (\gamma_E - \gamma_R) E[1(y_t + \beta_{t-1} > 0) - 1(y_t > 0) (y_t + \beta_{t-1}) | H_{t-1}].$$

The decomposition of the conditional response into a direct effect and an indirect effect generalizes to larger values of  $h$ . For instance, for  $h = 2$ ,

$$\begin{aligned} \text{CIRF}_2(H_{t-1}) &= E(y_{t+2}^* - y_{t+2} | H_{t-1}) \\ &= \underbrace{E[(\beta_{t+1}^* - \beta_{t+1}) x_{t+2} + (\gamma_{t+1}^* - \gamma_{t+1}) y_{t+1}^* | H_{t-1}]}_{\text{Indirect effect due to time } t+1 \text{ change in parameters}} \\ &\quad + \underbrace{E[\gamma_{t+1}(\beta_t^* - \beta_t) x_{t+1} + \gamma_{t+1}(\gamma_t^* - \gamma_t) y_t^* | H_{t-1}]}_{\text{Indirect effect due to time } t \text{ change in parameters}} \\ &\quad + \underbrace{E[\gamma_{t+1} \gamma_t \beta_{t-1} | H_{t-1}]}_{\text{Direct effect if no change in parameters}}, \end{aligned}$$

where the last term is the CIRF at  $h = 2$  derived in Proposition 3.1 under the assumption that  $\mathcal{H}^* = \mathcal{H}$ . This term captures the direct effect for  $h = 2$ . The indirect effect is represented by the first two terms. Characterizing these expectations analytically becomes intractable, even under strong assumptions about the conditional distribution.

The overall message is that when  $H_t$  depends on  $y_t$ , the conditional IRF is no longer the same as the one defined in Proposition 3.1. It now contains additional terms that capture the indirect effect of the shock in  $\varepsilon_{1t}$  on  $y_{t+h}$  that operates through the effect of the shock on the transition path of  $H_t$  through  $H_{t+h-1}$ .

## 4.2 Asymptotic bias in the LP estimator when $H_t$ is endogenous

We now investigate the effect of endogenizing  $H_t$  on the estimand of a state-dependent LP. For simplicity, we again focus on the simple bivariate model considered in (9) with  $H_t = 1(y_t > 0)$ . The state-dependent LP in this context is given by

$$y_{t+h} = b_{E,h} x_t H_{t-1} + \pi'_{E,h} W_{t-1} H_{t-1} + b_{R,h} x_t (1 - H_{t-1}) + \pi'_{R,h} W_{t-1} (1 - H_{t-1}) + v_{t+h}, \quad (10)$$

<sup>6</sup>Further simplifying this expression involves computing truncated moments of  $y_t + \beta_{t-1}$ , conditionally on  $H_{t-1}$ . This can be done for  $h = 1$  under parametric assumptions on the conditional distribution of  $y_t$  given  $H_{t-1}$ . However, this approach quickly becomes intractable as we increase the value of  $h$ . A simpler approach is to use numerical methods to approximate this expectation, which is the approach we use below to evaluate the asymptotic bias of the LP estimates.

where  $W'_{t-1} = (1, y_{t-1})$ .

Proceeding as in Section 3.2, we can show that the LP estimate of  $b_{E,h}$  converges in probability to

$$b_{E,h} = \frac{E(x_t H_{t-1} y_{t+h})}{E(x_t^2 H_{t-1})} = \frac{E(x_t y_{t+h} | H_{t-1} = 1)}{E(x_t^2 | H_{t-1} = 1)}.$$

The LS estimate of  $b_{E,h}$  in (10) can be obtained from an OLS regression of  $y_{t+h}$  on  $x_t$ , conditionally on  $H_{t-1} = 1$ . This result does not depend on whether  $H_t$  is exogenous or endogenous. Rather it follows from the fact that  $E(x_t H_{t-1} W_{t-1}) = 0$  by the m.d.s. assumption on  $\varepsilon_t$  (i.e., by Assumption 1).

When  $h = 0$ , one can easily show that  $b_{E,0} = \beta_E = CIRF_0(H_{t-1})$ . Thus, the state-dependent LP recovers the correct impact conditional IRF even when  $H_t$  depends on  $y_t$ . This follows because conditionally on  $H_{t-1} = 1$ , the structural model equation for  $y_t$  is

$$y_t = \beta_E x_t + \gamma_E y_{t-1} + \varepsilon_{2t},$$

and a linear local projection of  $y_t$  onto  $x_t$  and  $y_{t-1}$  that conditions on observations with  $H_{t-1} = 1$  recovers  $\beta_E$  provided  $\varepsilon_{2t}$  is orthogonal to  $x_t$  and  $y_{t-1}$  (conditionally on  $H_{t-1} = 1$ ). Assumption 1 alone suffices for this result.

When  $h = 1$ , evaluating  $b_{E,1} = p \lim \hat{b}_{E,1}$  is more challenging when  $H_t$  is endogenous. To see this, note that conditionally on  $H_{t-1} = 1$ , the equation for  $y_{t+1}$  can be described as

$$y_{t+1} = \gamma_t \beta_E x_t + \gamma_t \gamma_E y_{t-1} + u_{t+1},$$

where

$$u_{t+1} = \beta_t \varepsilon_{1t+1} + \gamma_t \varepsilon_{2t} + \varepsilon_{2t+1}.$$

Thus, conditionally on  $H_{t-1}$ , the model for  $y_{t+1}$  is state-dependent because it depends on  $H_t$  through the parameters  $\gamma_t$  and  $\beta_t$ .

As explained in Section 3.2, under exogeneity of  $H_t$  (as stated in Assumption 2), the LP estimand of the slope coefficient associated with  $x_t H_{t-1}$  is equal to  $E(\gamma_t | H_{t-1} = 1) \beta_E$ . This is the correct CIRF when  $H_t$  satisfies Assumption 2 and it is the direct effect contained in the population CIRF when  $H_t$  is endogenous. The LP estimand of this coefficient is not necessarily equal to  $E(\gamma_t | H_{t-1} = 1) \beta_E$ , when  $H_t = 1$  ( $y_t > 0$ ). The main reason is that Assumption 2 is not satisfied. For this choice,  $H_t$  and  $\varepsilon_t$  are no longer mean independent, conditionally on  $H_{t-1}$ . For instance,

$$E(\varepsilon_{1t} | H_t, \mathcal{F}^{t-1}) = E(\varepsilon_{1t} | y_t > 0, \mathcal{F}^{t-1}) = E(\varepsilon_{1t} | \beta_{t-1} \varepsilon_{1t} + \varepsilon_{2t} > -\gamma_{t-1} y_{t-1}, \mathcal{F}^{t-1}),$$

which is a conditional truncated moment of  $\varepsilon_{1t}$ . Although under Assumption 1,  $\varepsilon_{1t}$  has mean zero conditionally on  $\mathcal{F}^{t-1}$ , adding information on  $H_t$  in the form of the restriction  $\beta_{t-1} \varepsilon_{1t} + \varepsilon_{2t} > -\gamma_{t-1} y_{t-1}$

makes this mean not zero. The same is true for the second conditional moment of  $\varepsilon_t$ . For instance,  $E(\varepsilon_{1t}^2 | H_t, \mathcal{F}^{t-1})$  is no longer equal to  $\sigma_1^2 = E(\varepsilon_{1t} | \mathcal{F}^{t-1})$ .

Next, we evaluate the limit of the LP estimator by simulations and compare it with the population CIRF and its decomposition into a direct and indirect effect.

### 4.3 Does the LP Estimator Converge to the Population Response?

The literature has taken for granted that the LP estimator asymptotically recovers the population response, when the DGP is a state-dependent structural VAR model (see, e.g., Auerbach and Gorodnichenko (2013a); Alloza (2022)). Our analysis shows that this conclusion is indeed correct, when  $H_t$  is exogenous with respect to  $z_t$ . In the empirically more relevant case when  $H_t$  directly or indirectly depends on  $y_t$ , however, our sufficient conditions ensure the asymptotic validity of the state-dependent LP estimator only for the impact response. Although our theoretical analysis does not formally prove that the LP estimator of the response function is invalid at other horizons in this case, there is no presumption that it does recover the population response function. In this section, we explore this question based on several stylized bivariate DGPs and show that the LP estimator of the response function indeed appears to be inconsistent when  $H_t$  is endogenous. We consider four DGPs. The first three DGPs focus on the special case where  $x_t$  is a directly observed i.i.d. shock, whereas DGP4 considers the case where  $x_t$  is an AR(1) process. More specifically, we let

$$\begin{aligned} x_t &= \rho x_{t-1} + \varepsilon_{1t} \\ y_t &= \beta_{t-1} x_t + \alpha_{t-1} x_{t-1} + \gamma_{t-1} y_{t-1} + \varepsilon_{2t} \end{aligned} \tag{11}$$

where

$$\begin{aligned} \alpha_{t-1} &= \alpha_E H_{t-1} + \alpha_R (1 - H_{t-1}) \\ \beta_{t-1} &= \beta_E H_{t-1} + \beta_R (1 - H_{t-1}), \\ \gamma_{t-1} &= \gamma_E H_{t-1} + \gamma_R (1 - H_{t-1}), \end{aligned} \tag{12}$$

$\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})' \sim N(0, I_2)$  and  $H_t$  is an indicator function that determines the state of the economy. When  $H_t = 1$  the economy is in an expansion,  $E$ , and when  $H_t = 0$  the economy is in recession,  $R$ .

DGP1-DGP3 focus on the case where  $x_t = \varepsilon_{1t}$ , and therefore set  $\rho = 0$ . In addition, these DGPs set  $\alpha_{t-1} = 0$ , so that only  $x_t$  and  $y_{t-1}$  enter the equation for  $y_t$ . In DGP1,  $H_t = F(q_t) \equiv 1(q_t > 0)$ , where  $q_t$  follows an exogenous process

$$q_t = 0.6q_{t-1} + u_t,$$

and  $u_t \sim N(0, 1)$  is independent of  $\varepsilon_t$ . DGP2 and DGP3 differ from DGP1 in that the indicator function is given by  $H_t = 1(y_t > 0)$ , so that the state of the economy is determined endogenously. DGP1 and DGP2 set  $\beta_E = 2.4$ ,  $\beta_R = 1.6$ ,  $\gamma_E = 0.7$  and  $\gamma_R = 0.1$ , whereas DGP3 sets  $\beta_E = 2.5$ ,  $\beta_R = 3.5$ ,  $\gamma_E = 0.9$  and  $\gamma_R = -0.1$ . Finally, DGP4 specifies  $x_t$  as an AR(1) process with  $\rho = 0.8$ . In addition, this DGP sets  $\alpha_{t-1} \neq 0$ , with  $\alpha_E = 1.2$  and  $\alpha_R = 0.9$ .

We consider the effect on  $y_{t+h}$  of a shock of size 1 in  $\varepsilon_{1t}$ . The conditional impulse response function is evaluated as  $E(y_{t+h}^* - y_{t+h} | H_{t-1})$ , whereas the LP estimands are evaluated as (10). We also compute the direct effect (given by the formula in Proposition 3.1) and the indirect effect (which we obtain as the difference between  $E(y_{t+h}^* - y_{t+h} | H_{t-1})$  and the direct effect). The number of draws used to compute all these conditional expectations is equal to 50 million. In addition to reporting results of the effect of the shock on the level of  $y_{t+h}$ , we also compute the cumulative effects. These are obtained by summing the individual CIRFs and the corresponding LP coefficients. For instance, the cumulative CIRF at horizon  $h = 1$  equals  $\sum_{h=0}^1 E(y_{t+h}^* - y_{t+h} | H_{t-1})$  and the LP estimand is  $\sum_{h=0}^1 b_{i,h}$  with  $b_{i,h} = b_{E,h}$  if  $H_{t-1} = 1$  and  $b_{i,h} = b_{R,h}$  if  $H_{t-1} = 0$ .

Figures 1 and 2 contain the results when  $H_t$  is exogenous (DGP1) whereas Figures 3 through 8 contain results for the endogenous case (DGP2, DGP3 and DGP4). Starting with DGP1, Figure 1 shows that the CIRF is equal to the LP estimand at all horizons. In addition, the indirect effect is zero, making the CIRF equal to the direct effect. This is consistent with our theoretical results (cf. Proposition 3.2). Because the LP estimand coincides with the CIRF for  $y_{t+h}$ , LP also recovers the cumulative effect, as shown by Figure 2.

Figures 3 through 8 show that these results change when  $H_t = 1(y_t > 0)$ , making  $H_t$  endogenous with respect to  $\varepsilon_{1t}$ . These figures show that the LP estimands no longer coincide with the population response function of interest (both in levels and as a sum). In particular, although the impact effect is recovered by the state-dependent LP, this is no longer true at intermediate values of  $h$ . As  $h$  increases further, the CIRF and the corresponding LP estimand both tend to zero, making the bias disappear. This is no longer true for the cumulative LP bias, however, which remains non-zero for all values of  $h$ . The decomposition of the CIRF into the direct and indirect effect shows that the LP estimand follows closely the direct effect, while missing the indirect effect.

The size of the asymptotic bias depends on the parameter values we choose. In DGP2 and DGP3, the bias increases with  $\gamma_E - \gamma_R$ , implying that it is larger in absolute value in DGP3 than in DGP2. For example, compare Figures 3 and 5 for the CIRF and Figures 4 and 6 for the cumulative CIRF. Although Figures 3 and 5 seem to suggest that the bias of the LP estimator is modest relative to the value of the impulse response function, this bias is significant when measured as a function of the

population CIRF of interest. For instance, for DGP2, the bias of LP relative to the CIRF is equal  $-10\%$ ,  $-13\%$ ,  $-14\%$ , and  $-15\%$  for  $h = 1, 2, 3, 4$ , when in expansions, and  $-20\%$ ,  $-20\%$ ,  $-20\%$ ,  $-21\%$  when in recessions. These numbers imply a relative bias for the cumulative response function that varies between  $-4\%$  and  $-7\%$  in expansions and  $-6\%$  and  $-10\%$  in recessions. For DGP3, the relative bias of the LP estimator for the CIRF varies between  $-6\%$  and  $-14\%$  in expansions and  $-29\%$  and  $-40\%$  in recessions. This translates to a relative bias in the cumulative CIRF that varies between  $-3\%$  and  $-8\%$  for expansions and  $-11\%$  and  $-23\%$  for recessions.

The results for DGP4 follow the same patterns for DGP2 and DGP3, although in relative terms the bias is smaller than in DGP2 and DGP3. In particular, the relative bias of LP relative to the CIRF is at most  $-5.6\%$  in expansions and  $-10.4\%$  in recessions. For the cumulative effects, the maximum relative bias over  $h = 1, \dots, 10$  is  $-3.6\%$  for expansions and  $-7.4\%$  in recessions.

It is standard in nonlinear time series analysis to report responses to one and two standard deviation shocks in recognition of the fact that in nonlinear models changing the magnitude of the shock may affect the value of the impulse responses. As we have shown, the state-dependent LP estimator implicitly sets the shock size  $\delta$  to unity, which need not correspond to a one standard deviation shock in general. More generally, for a shock of size  $\delta$  the LP estimator may be scaled by a factor of  $\delta$ . This approach, of course, is only expected to work when  $H_t$  is exogenous. It is useful to illustrate the sensitivity of the asymptotic bias of the LP estimator to the magnitude of the  $\varepsilon_{1t}$  shock when  $H_t$  is endogenous. In our DGPs, setting  $\delta = 1$  corresponds to a one standard deviation shock. Figure 9 illustrates that the asymptotic bias of the LP estimator relative to the population CIRF increases substantially when increasing  $\delta$  from 1 to 2, corresponding to a two standard deviation shock. In this example, which is based on DGP 4, at most horizons, the asymptotic bias is near 10 percent in expansions and may exceed 20 percent in recessions.

## 5 Conclusion

State-dependent LP impulse response estimators have become one of the most commonly used tools in empirically macroeconomics in recent years. The idea that the effects of economic shocks may differ depending on the state of the economy has a long tradition, but apparent nonlinearities in recent macroeconomic data such as the zero lower bound on interest rates have, if anything, further heightened interest among applied researchers in such state dependencies. Much of what we know about the state dependence of fiscal multipliers and the state-dependent effects of monetary policy shocks, for example, is based on this LP approach, yet the validity of this approach has never been formally established.

Although there has been much discussion of the perceived advantages of this approach in the literature compared to the estimation of state-dependent structural VAR models, including its apparent simplicity and its potential robustness to possible dynamic misspecification of nonlinear VAR models, it is not clear under what conditions the state-dependent LP estimator recovers the population impulse response functions of interest. It also remains unclear what impulse response function the LPs are estimating, among many competing impulse response concepts. In this paper, we made precise the nature of the state-dependent impulse responses captured by the LP estimator, and we provided sufficient conditions, under which this estimator is consistent. These conditions tend to be violated in many empirical studies. Our analysis suggests that, when the state of the economy is endogenously determined, the LP estimator tends to be valid only for the impact response. This is a concern not only for impulse response analysis but also for the construction of fiscal and monetary multipliers that are often computed at higher horizons (or relative to the peak in the response function).

While our theoretical analysis does not formally establish the inconsistency of the LP estimator (and the multipliers derived from those responses) when the state of the economy is endogenous, we showed numerically that in practice the LP estimator of the response function tends to be asymptotically biased. The fact that many applications of the state-dependent LP estimator implicitly treat the state of the economy as exogenous with respect to the model variables, when it clearly is endogenous, calls into question their substantive conclusions. This result is important not only from an econometric point of view, but also for the ongoing debate about the magnitude of fiscal and monetary multipliers. Our analysis highlights the need to be specific about nature of the state dependence when applying the state-dependent LP estimator. LP estimators cannot be interpreted and their validity cannot be assessed without taking a stand on the data generating process.

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## A Appendix

**Proof of Proposition 3.1.** The proof for  $h = 0$  and  $h = 1$  is in the text. We omit the proof for general  $h$  since it follows from similar arguments.

**Proof of Proposition 3.2.** To define  $\hat{b}_{E,h}$ , let

$$Y_{T \times k} = \begin{pmatrix} y'_{1+h} \\ \vdots \\ y'_{T+h} \end{pmatrix}, \quad X_1_{T \times 1} = \begin{pmatrix} x_1 H_0 \\ \vdots \\ x_T H_{T-1} \end{pmatrix}, \quad \text{and} \quad X_2_{T \times (np+1)} = \begin{pmatrix} W'_0 H_0 \\ \vdots \\ W'_{T-1} H_{T-1} \end{pmatrix},$$

and define  $M_2 = I_T - X_2 (X'_2 X_2)^{-1} X'_2$ .

By the Frisch-Waugh-Lovell (FWL) Theorem,  $\hat{b}'_{E,h} = (X'_1 M_2 X_1)^{-1} X'_1 M_2 Y$ , or

$$\hat{b}_{E,h} = T^{-1} (Y' M_2 X_1) (T^{-1} X'_1 M_2 X_1)^{-1} \equiv \hat{Q}_{1y,2,h} \hat{Q}_{11,2}^{-1}.$$

A similar expression holds for  $\hat{b}_{R,h}$  with the difference that the regressors  $x_t$  and  $W_{t-1}$  are interacted with  $1 - H_{t-1}$  rather than  $H_{t-1}$ .

Our goal is to derive the probability limit of  $\hat{b}_{E,h}$  (and  $\hat{b}_{R,h}$ ) as  $T \rightarrow \infty$ . We can write

$$\begin{aligned} \hat{Q}_{11,2} &= T^{-1} X'_1 X_1 - T^{-1} X'_1 X_2 (T^{-1} X'_2 X_2)^{-1} T^{-1} X'_2 X_1, \quad \text{and} \\ \hat{Q}_{1y,2,h} &= T^{-1} Y' X_1 - T^{-1} Y' X_2 (T^{-1} X'_2 X_2)^{-1} T^{-1} X'_2 X_1. \end{aligned}$$

If a law of large numbers applies to each term<sup>7</sup>,

$$\begin{aligned} \hat{Q}_{11,2} &\xrightarrow{P} Q_{11,2} \equiv E(x_t^2 H_{t-1}) - E(x_t H_{t-1} W'_{t-1}) [E(W_{t-1} W'_{t-1} H_{t-1})]^{-1} E(W_{t-1} H_{t-1} x_t), \quad \text{and} \\ \hat{Q}_{1y,2,h} &\xrightarrow{P} Q_{1y,2,h} \equiv E(y_{t+h} x_t H_{t-1}) - E(y_{t+h} H_{t-1} W'_{t-1}) [E(W_{t-1} W'_{t-1} H_{t-1})]^{-1} E(W_{t-1} H_{t-1} x_t). \end{aligned}$$

We distinguish two cases: (i)  $x_t = \varepsilon_{1t}$ , and (ii)  $x_t = \mu_{1,t-1} + B_{11,t-1}(L) x_{t-1} + B_{12,t-1}(L) y_{t-1} + \varepsilon_{1t} = \alpha'_{t-1} W_{t-1} + \varepsilon_{1t}$  (where  $\alpha_{t-1}$  is a state-dependent vector that collects the coefficients of  $\mu_{1,t-1}$ ,  $B_{11,t-1}(L)$  and  $B_{12,t-1}(L)$ ).

<sup>7</sup>We assume that the data are strictly stationary and ergodic and that the usual moment and rank conditions on the regressors are satisfied. We leave these as implicit high level assumptions since our focus here is on the conditions that  $H_t$  needs to satisfy in order for the LP estimator to be consistent. Kole and van Dijk (2021) (and references therein) provide primitive conditions for stationarity and ergodicity of a Markov Switching SVAR model when the states  $H_t$  are assumed to be a first-order exogenous Markov process. Deriving analogous primitive conditions for our setting, when the process for the exogenous  $H_t$  is not specified, is beyond the scope of this paper.

In case (i), it is easy to see that  $E(x_t H_{t-1} W'_{t-1}) = 0$  under the assumption that  $x_t = \varepsilon_{1t}$  is a m.d.s. Thus,

$$Q_{11.2} = E(x_t^2 H_{t-1}) \text{ and } Q_{1y.2} = E(y_{t+h} x_t H_{t-1}),$$

implying that<sup>8</sup>

$$\hat{b}_{E,h} \xrightarrow{p} b_{E,h} \equiv E(y_{t+h} x_t H_{t-1}) [E(x_t^2 H_{t-1})]^{-1} = E(y_{t+h} x_t | H_{t-1} = 1) [E(x_t^2 | H_{t-1} = 1)]^{-1}.$$

In case (ii), we can show that

$$\begin{aligned} Q_{11.2} &= E(\varepsilon_{1t}^2 H_{t-1}) = \Pr(H_{t-1} = 1) E(\varepsilon_{1t}^2 | H_{t-1} = 1) \text{ and} \\ Q_{1y.2} &= E(y_{t+h} \varepsilon_{1t} H_{t-1}) = \Pr(H_{t-1} = 1) E(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1), \end{aligned}$$

implying that  $b_{E,h} = E(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1) [E(\varepsilon_{1t}^2 | H_{t-1} = 1)]^{-1}$ . Heuristically, this follows because by the FWL theorem, and conditioning on  $H_{t-1} = 1$ , the slope coefficient associated with  $x_t$  from regressing  $y_{t+h}$  on  $x_t$  and  $W_{t-1}$  can be obtained in two steps. First, we regress  $x_t$  on  $W_{t-1}$  (interacted with  $H_{t-1}$ ) and obtain the residual. Under our identification condition, this is  $\varepsilon_{1t}$ . Then, we regress  $y_{t+h}$  on  $\varepsilon_{1t}$  (interacted with  $H_{t-1}$ ). More specifically, note that

$$E(x_t H_{t-1} W'_{t-1}) = E(\alpha'_{t-1} W_{t-1} W'_{t-1} H_{t-1}) + E(\varepsilon_{1t} H_{t-1} W'_{t-1}) = E(\alpha'_{t-1} W_{t-1} W'_{t-1} H_{t-1}),$$

since  $E(\varepsilon_{1t} H_{t-1} W'_{t-1}) = 0$  by the m.d.s. assumption on  $\varepsilon_{1t}$ . It follows that

$$E(x_t H_{t-1} W'_{t-1}) = \alpha'_E E(W_{t-1} W'_{t-1} | H_{t-1} = 1) \Pr(H_{t-1} = 1).$$

Hence, the term  $E(x_t H_{t-1} W'_{t-1}) [E(W_{t-1} W'_{t-1} H_{t-1})]^{-1} E(W_{t-1} H_{t-1} x_t)$  equals

$$\begin{aligned} &\alpha'_E E(W_{t-1} W'_{t-1} | H_{t-1} = 1) [E(W_{t-1} W'_{t-1} | H_{t-1} = 1)]^{-1} E(W_{t-1} W'_{t-1} | H_{t-1} = 1) \alpha_E \Pr(H_{t-1} = 1) \\ &= \alpha'_E E(W_{t-1} W'_{t-1} | H_{t-1} = 1) \alpha_E \Pr(H_{t-1} = 1) \\ &= E(\alpha'_{t-1} W_{t-1} W'_{t-1} \alpha_{t-1} | H_{t-1} = 1) \Pr(H_{t-1} = 1). \end{aligned}$$

Since  $x_t^2 = (\alpha'_{t-1} W_{t-1} + \varepsilon_{1t})^2 = \alpha'_{t-1} W_{t-1} W'_{t-1} \alpha_{t-1} + 2\alpha'_{t-1} W_{t-1} \varepsilon_{1t} + \varepsilon_{1t}^2$ , where the second term has conditional mean equal to zero, it follows that

$$Q_{11.2} = \Pr(H_{t-1} = 1) E(\varepsilon_{1t}^2 | H_{t-1} = 1).$$

We can use similar arguments to show that

$$Q_{1y.2} = \Pr(H_{t-1} = 1) E(y_{t+h} \varepsilon_{1t} | H_{t-1} = 1).$$

---

<sup>8</sup>This result is consistent with the fact that when  $x_t$  is a directly observed shock we can simply regress  $y_{t+h}$  onto  $x_t H_{t-1}$  to obtain a consistent estimator of  $b_{E,h}$ . When  $x_t = \varepsilon_{1t}$ , adding the controls  $W_{t-1} H_{t-1}$  is not required for consistency, but can be important for efficiency.

Thus, both in cases (i) and (ii), we conclude that

$$\hat{b}_{E,h} \xrightarrow{p} b_{E,h} = E(y_{t+h}\varepsilon_{1t} | H_{t-1} = 1) [E(\varepsilon_{1t}^2 | H_{t-1} = 1)]^{-1} \equiv \mathcal{N}_h \mathcal{D},$$

where  $\mathcal{N}_h$  stands for numerator and  $\mathcal{D}$  is the denominator. Next, we express  $\mathcal{N}_h$  and  $\mathcal{D}$  in terms of the model's parameters. To evaluate  $\mathcal{N}_h$ , we use the fact that for any  $h$ ,  $y_{t+h} = \mathbb{S}_k Z_{t+h}$ , where  $Z_{t+h}$  is obtained from the companion form representation of the model given by (5). Consider first  $h = 0$ . Then

$$Z_t = a_{t-1} + A_{t-1}Z_{t-1} + \xi_t,$$

where

$$\xi_t = \begin{pmatrix} \eta_t \\ 0 \end{pmatrix} = \begin{pmatrix} C_{t-1}^{-1}e_{1n}\varepsilon_{1t} + C_{t-1}^{-1}I_{2:n}\varepsilon_{2t} \\ 0 \end{pmatrix} = (e_{1p} \otimes C_{t-1}^{-1}e_{1n})\varepsilon_{1t} + e_{1p} \otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t},$$

given that  $\eta_t = C_{t-1}^{-1}\varepsilon_t$  and  $\varepsilon_t = C_{t-1}^{-1}e_{1n}\varepsilon_{1t} + C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}$ , where  $e_{1n}$  and  $I_{2:n}$  are as defined in Section 3.1. Hence,

$$y_t = \mathbb{S}_k Z_t = \mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}e_{1n})\varepsilon_{1t} + \mathbb{S}_k(a_{t-1} + A_{t-1}Z_{t-1}) + \mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}). \quad (13)$$

Using the above decomposition of  $y_t$ , we can write  $\mathcal{N}_0 = E(y_t\varepsilon_{1t} | H_{t-1} = 1) = \mathcal{N}_{0,1} + \mathcal{N}_{0,2} + \mathcal{N}_{0,3}$ , where

$$\begin{aligned} \mathcal{N}_{0,1} &= E[\mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}e_{1n})\varepsilon_{1t}^2 | H_{t-1} = 1], \\ \mathcal{N}_{0,2} &= E[\mathbb{S}_k(a_{t-1} + A_{t-1}Z_{t-1})\varepsilon_{1t} | H_{t-1} = 1], \text{ and} \\ \mathcal{N}_{0,3} &= E[\mathbb{S}_k(e_{1p} \otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t})\varepsilon_{1t} | H_{t-1} = 1]. \end{aligned}$$

Under Assumption 1 and applying repeatedly the law of iterated expectations (LIE), it can be shown that  $\mathcal{N}_{0,2} = \mathcal{N}_{0,3} = 0$ , implying that  $\mathcal{N}_0 \equiv E(y_t\varepsilon_{1t} | H_{t-1} = 1) = \mathcal{N}_{0,1}$ . Thus,

$$\mathcal{N}_0 = \mathbb{S}_k(e_{1p} \otimes C_E^{-1}e_{1n})E(\varepsilon_{1t}^2 | H_{t-1} = 1).$$

Since  $b_{E,0} \equiv \mathcal{N}_0 \mathcal{D}$ , where  $\mathcal{D} \equiv [E(\varepsilon_{1t}^2 | H_{t-1} = 1)]^{-1}$ , this implies the result. A similar argument shows that

$$\hat{b}_{R,0} \xrightarrow{p} b_{R,0} = \mathbb{S}_k(e_{1p} \otimes C_R^{-1}e_{1n}).$$

These results show that the state-dependent LP regression (7) recovers the conditional IRF obtained in Proposition 3.1 with  $h = 0$  under Assumption 1. No further assumptions are required (provided a law of large numbers can be applied to  $\hat{Q}_{11,2}$  and  $\hat{Q}_{1y,2,0}$ ). In particular, conditional homoskedasticity of  $\varepsilon_t$  is not required. Nor do we need to impose further restrictions on the process driving state dependence.

As we will show next, this is no longer the case when  $h > 0$ . To illustrate this, consider  $h = 1$ . Now,

$$\hat{b}_{E,1} \xrightarrow{p} b_{E,1} \equiv E(y_{t+1}\varepsilon_{1t}|H_{t-1}=1) [E(\varepsilon_{1t}^2|H_{t-1}=1)]^{-1} \equiv \mathcal{N}_1 \mathcal{D}.$$

To obtain  $\mathcal{N}_1$ , we can use the fact that

$$\begin{aligned} y_{t+1} &= \mathbb{S}_k Z_{t+1} = \mathbb{S}_k(a_t + A_t Z_t + \xi_{t+1}) \\ &= \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1} + \xi_t) + \xi_{t+1}) \\ &= \mathbb{S}_k A_t \xi_t + \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1})) + \mathbb{S}_k \xi_{t+1}, \end{aligned} \quad (14)$$

where  $\xi_s = (e_{1p} \otimes C_{s-1}^{-1} e_{1n}) \varepsilon_{1s} + e_{1p} \otimes C_{s-1}^{-1} I_{2:n} \varepsilon_{2s}$  for  $s = t, t+1$ . This implies that  $\mathcal{N}_1 \equiv E(y_{t+1}\varepsilon_{1t}|H_{t-1}=1) = \mathcal{N}_{1,1} + \mathcal{N}_{1,2} + \mathcal{N}_{1,3}$ , where

$$\begin{aligned} \mathcal{N}_{1,1} &= E(\mathbb{S}_k A_t \xi_t \varepsilon_{1t} | H_{t-1} = 1), \\ \mathcal{N}_{1,2} &= E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1})) \varepsilon_{1t} | H_{t-1} = 1], \text{ and} \\ \mathcal{N}_{1,3} &= E[\mathbb{S}_k \xi_{t+1} \varepsilon_{1t} | H_{t-1} = 1]. \end{aligned}$$

Given the definition of  $\xi_{t+1}$ , we can easily see that  $\mathcal{N}_{1,3} = 0$  by Assumption 1, since it implies that  $E(\xi_{t+1} | \mathcal{F}^t) = 0$ . However, to conclude that  $\mathcal{N}_{1,2} = 0$ , we need further assumptions. More specifically, this term now depends on  $H_t$  (through  $a_t \equiv a_E H_t + a_R(1 - H_t)$  and  $A_t \equiv A_E H_t + A_R(1 - H_t)$ ). Conditionally on  $\mathcal{F}^{t-1}$ ,  $H_t$  and  $\varepsilon_{1t}$  may be correlated, implying that  $\mathcal{N}_{1,2}$  may be non-zero. Indeed, by the LIE, we can write

$$\mathcal{N}_{1,2} = E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1})) E(\varepsilon_{1t} | \mathcal{F}^{t-1}, H_t) | H_{t-1} = 1].$$

A sufficient condition for  $\mathcal{N}_{1,2} = 0$  is that  $E(\varepsilon_{1t} | \mathcal{F}^{t-1}, H_t) = 0$ , which holds under Assumptions 1 and 2(a) with  $h = 1$ . Under this condition,  $\mathcal{N}_1 = \mathcal{N}_{1,1}$ .

Additional conditions are also required to simplify  $\mathcal{N}_{1,1} = E(\mathbb{S}_k A_t \xi_t \varepsilon_{1t} | H_{t-1} = 1)$  and show that  $b_{E,1} \equiv \mathcal{N}_1 \mathcal{D} = CIRF_1(H_{t-1} = 1) \equiv E[\mathbb{S}_k A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}) | H_{t-1} = 1]$ . Using the definition of  $\xi_t$ ,  $\mathcal{N}_{1,1}$  can be decomposed as follows:

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}) \varepsilon_{1t}^2 | H_{t-1} = 1] + E[\mathbb{S}_k A_t (e_{1p} \otimes C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \varepsilon_{1t}) | H_{t-1} = 1].$$

The presence of  $A_t$  (which depends on  $H_t$ ) again complicates the evaluation of these expectations. For instance, the second term is not zero if  $E(\varepsilon_{1t} \varepsilon_{2t} | H_t, \mathcal{F}^{t-1}) \neq 0$  even if  $\Sigma$  is diagonal. Assumption 2(b) with  $h = 1$  ensures  $E(\varepsilon_{1t} \varepsilon_{2t} | H_t, \mathcal{F}^{t-1}) = 0$ , implying that

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t (e_{1p} \otimes C_{t-1}^{-1} e_{1n}) \varepsilon_{1t}^2 | H_{t-1} = 1].$$

It follows that

$$b_{E,1} = \frac{E[\mathbb{S}_k A_t(e_{1p} \otimes C_{t-1}^{-1} e_{1n}) \varepsilon_{1t}^2 | H_{t-1} = 1]}{E(\varepsilon_{1t}^2 | H_{t-1} = 1)}.$$

A sufficient condition for  $b_{E,1}$  to equal  $E[\mathbb{S}_k A_t(e_{1p} \otimes C_{t-1}^{-1} e_{1n}) | H_{t-1} = 1]$  is the conditional homoskedasticity condition  $E(\varepsilon_{1t}^2 | H_t, \mathcal{F}^{t-1}) = \sigma_1^2 = E(\varepsilon_{1t}^2 | \mathcal{F}^{t-1})$ . This is Assumption 2(b) with  $h = 1$ , which together with Assumption 1 and 2(b) ensures the consistency of the LP estimator for  $h = 1$ . The proof for other values of  $h$  follows from similar arguments provided Assumption 1 is strengthened by Assumption 2.

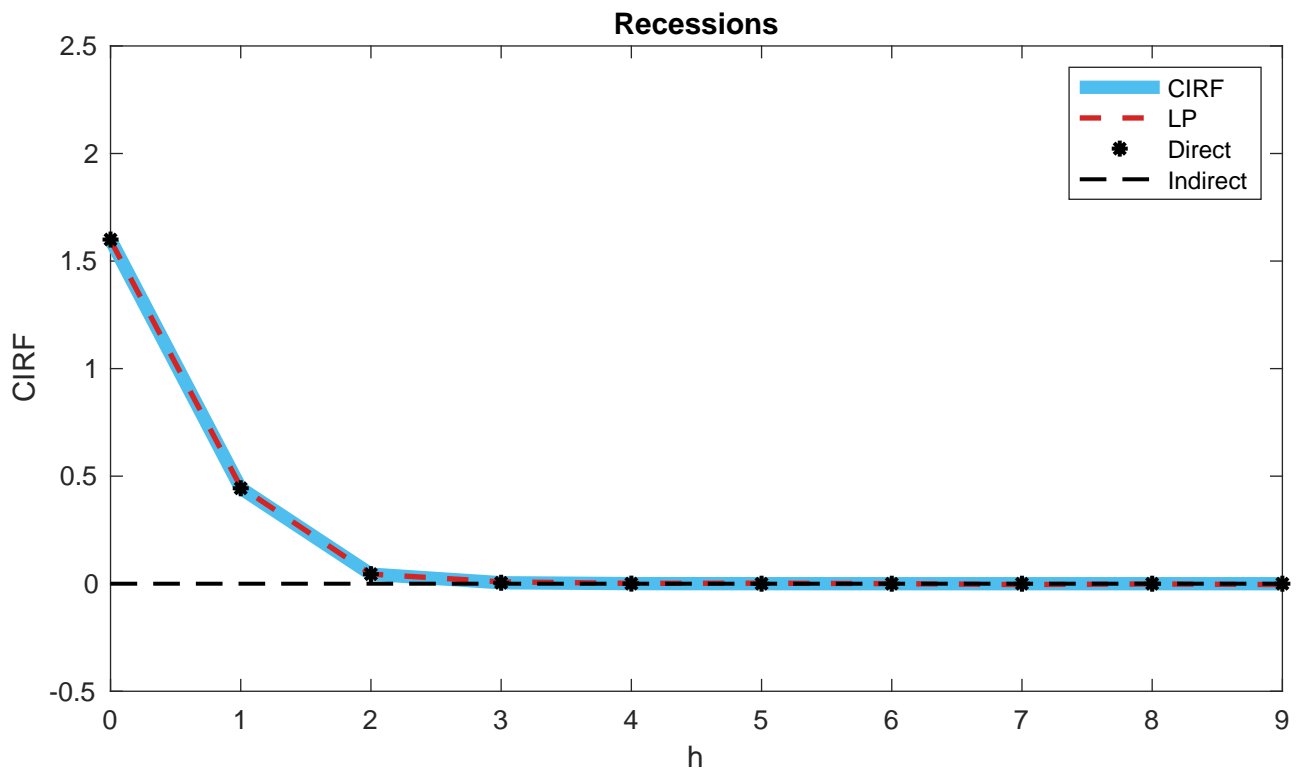
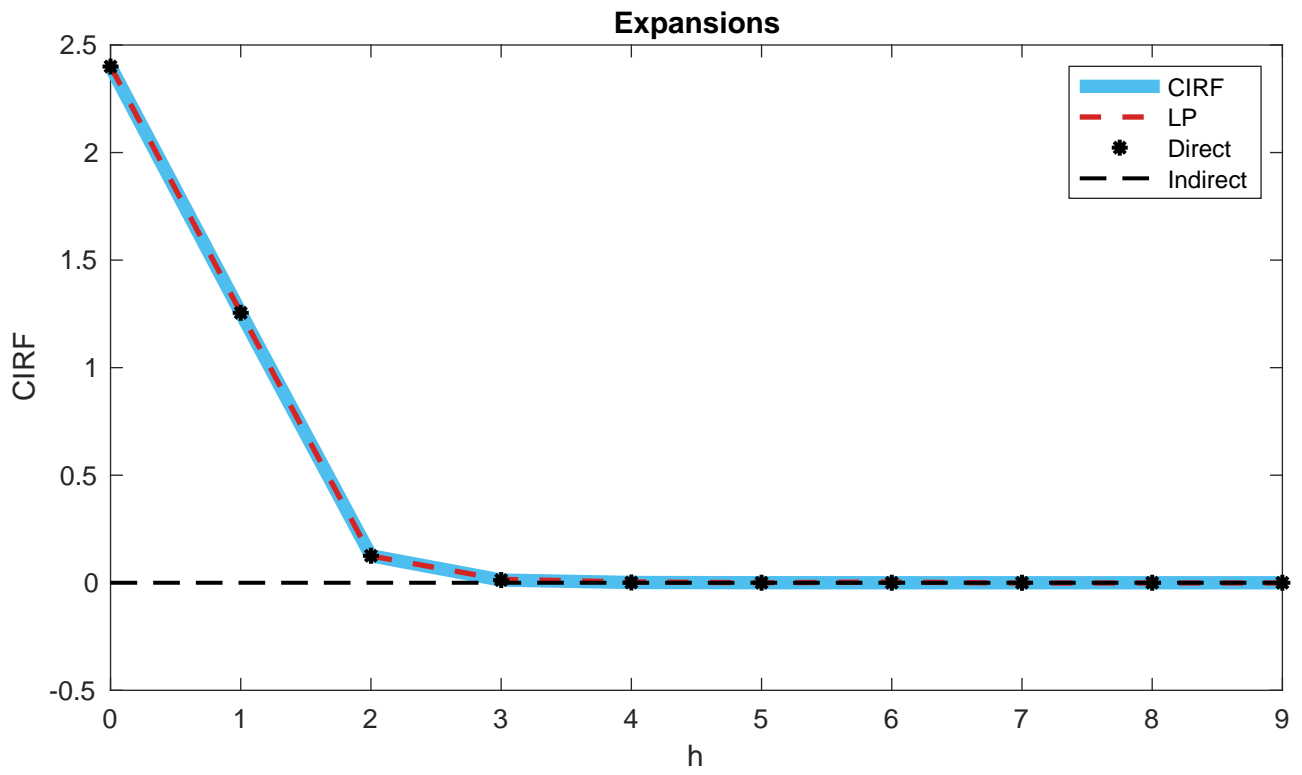


Figure 1: DGP1: Exogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Level Effects



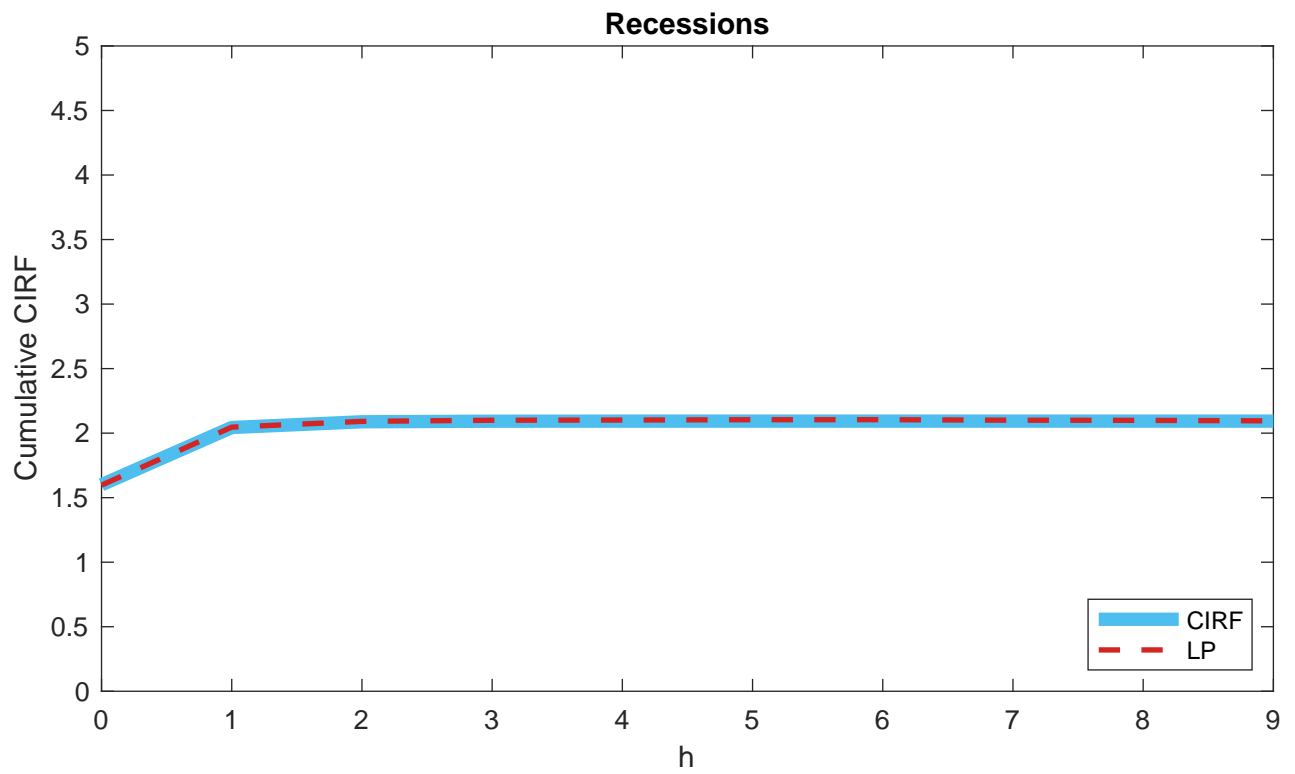
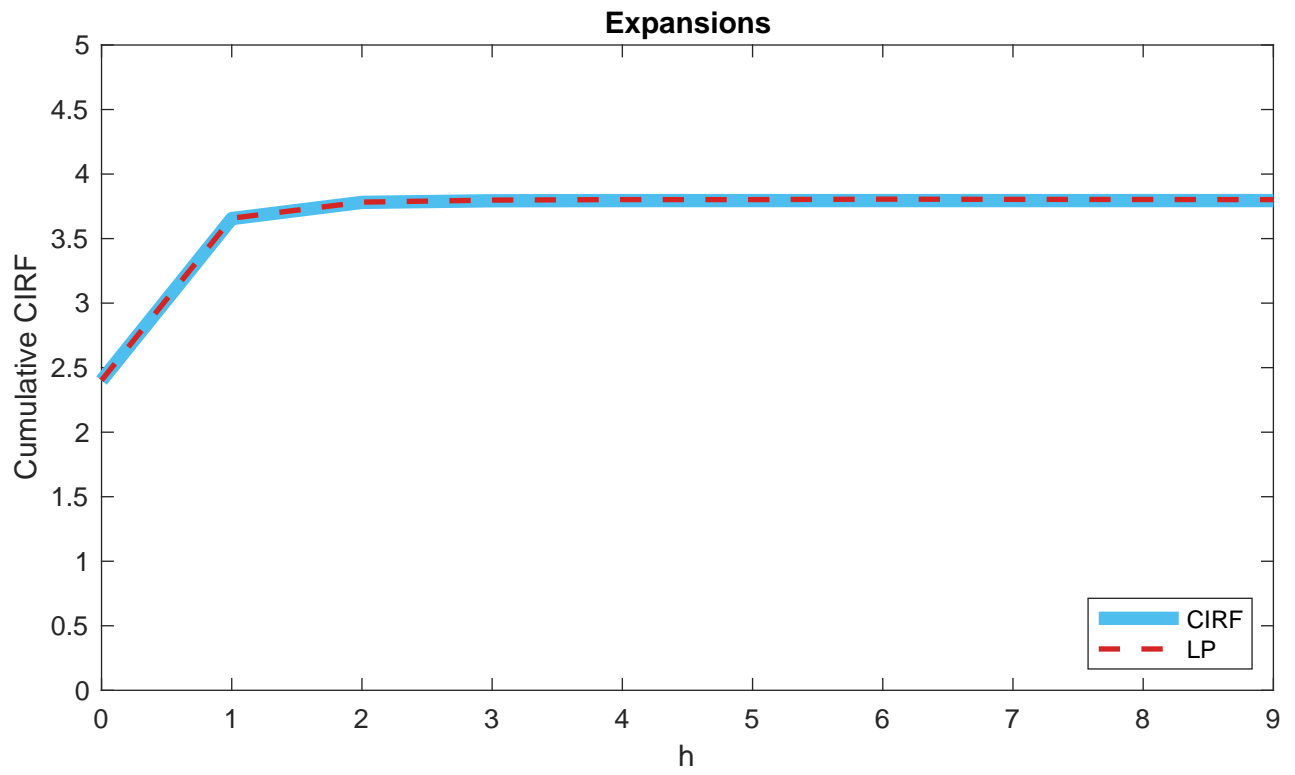


Figure 2: DGP1: Exogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Cumulative Effects

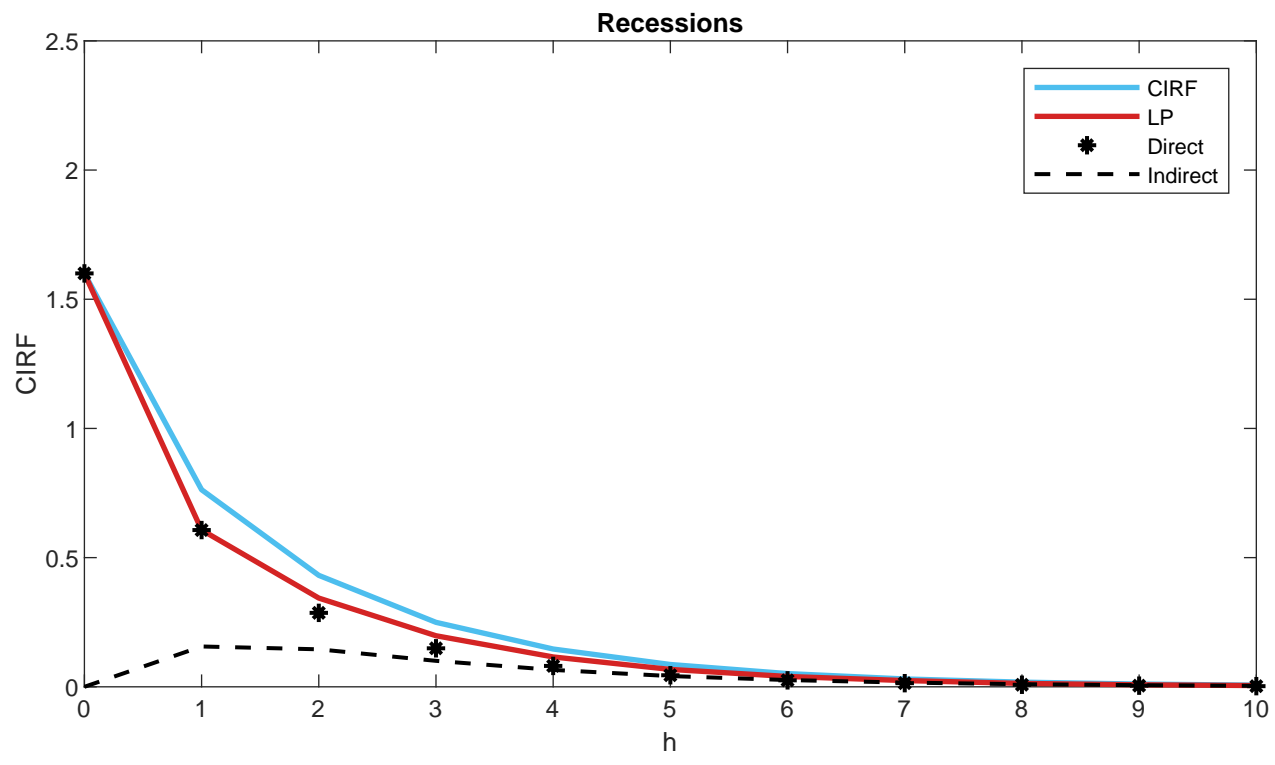
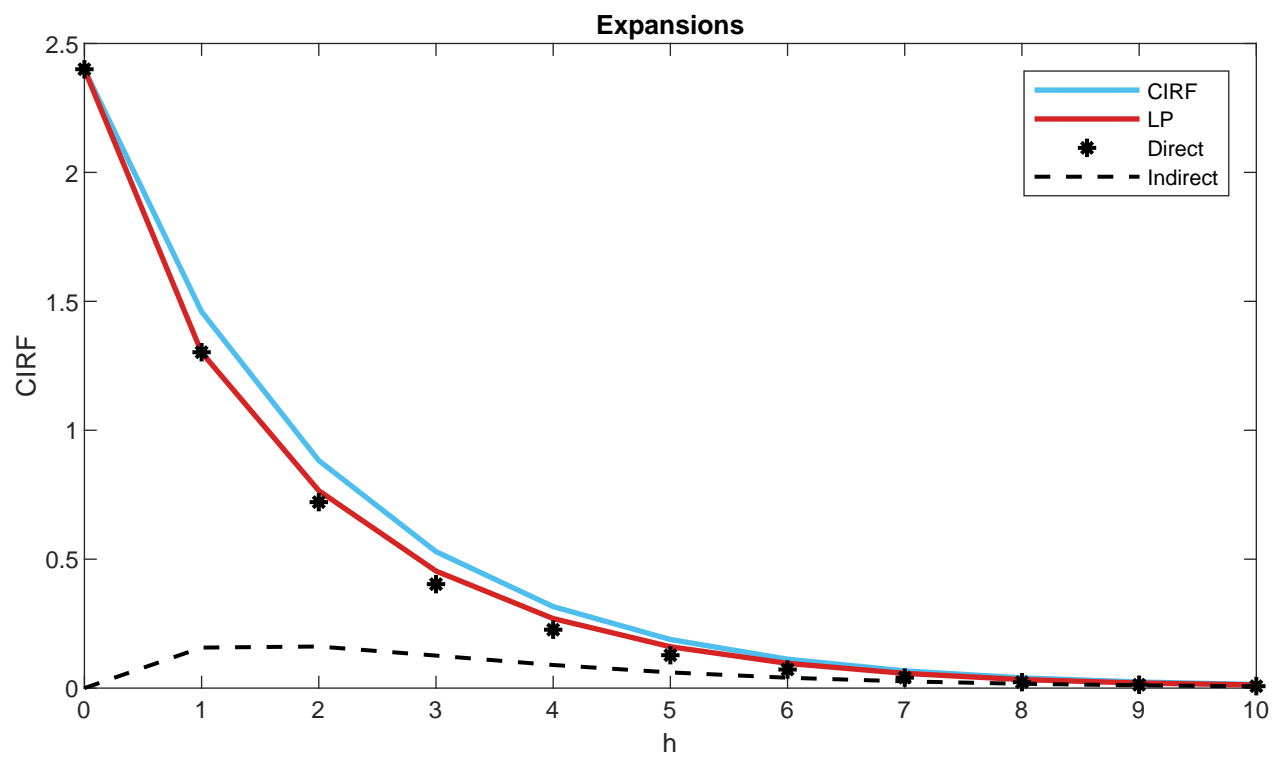


Figure 3: DGP2: Endogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Level Effects

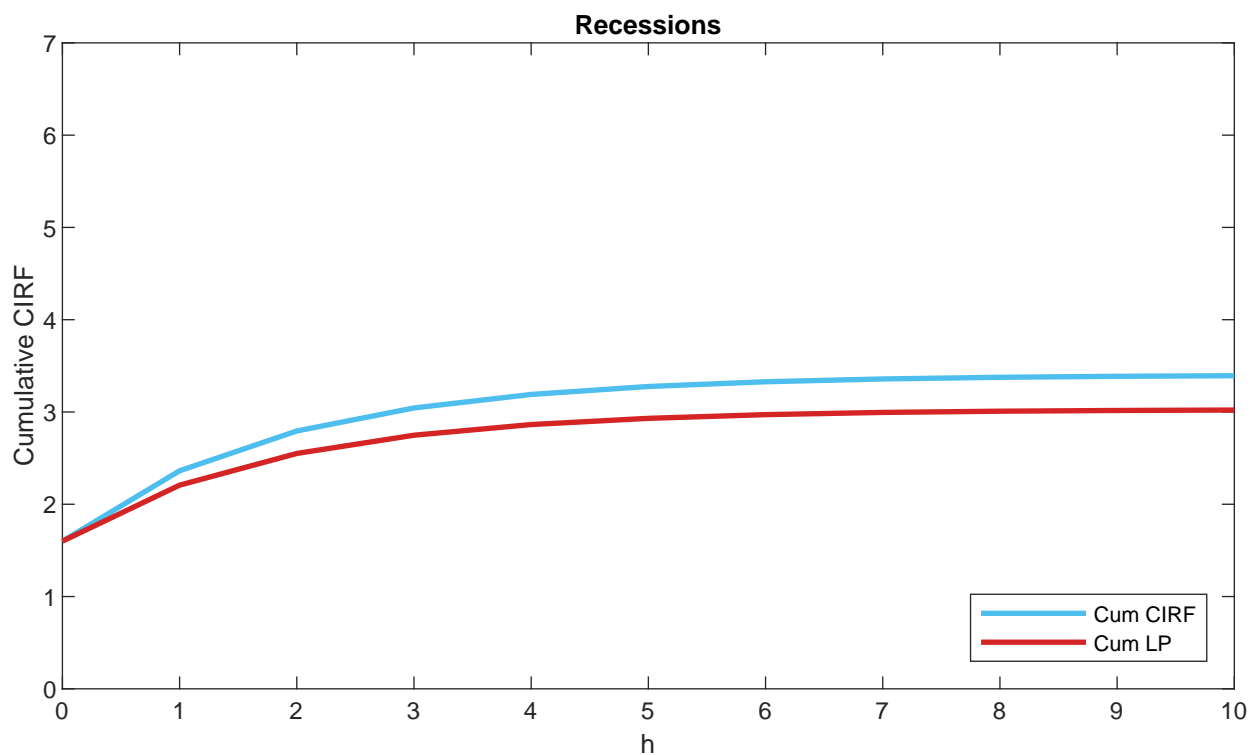
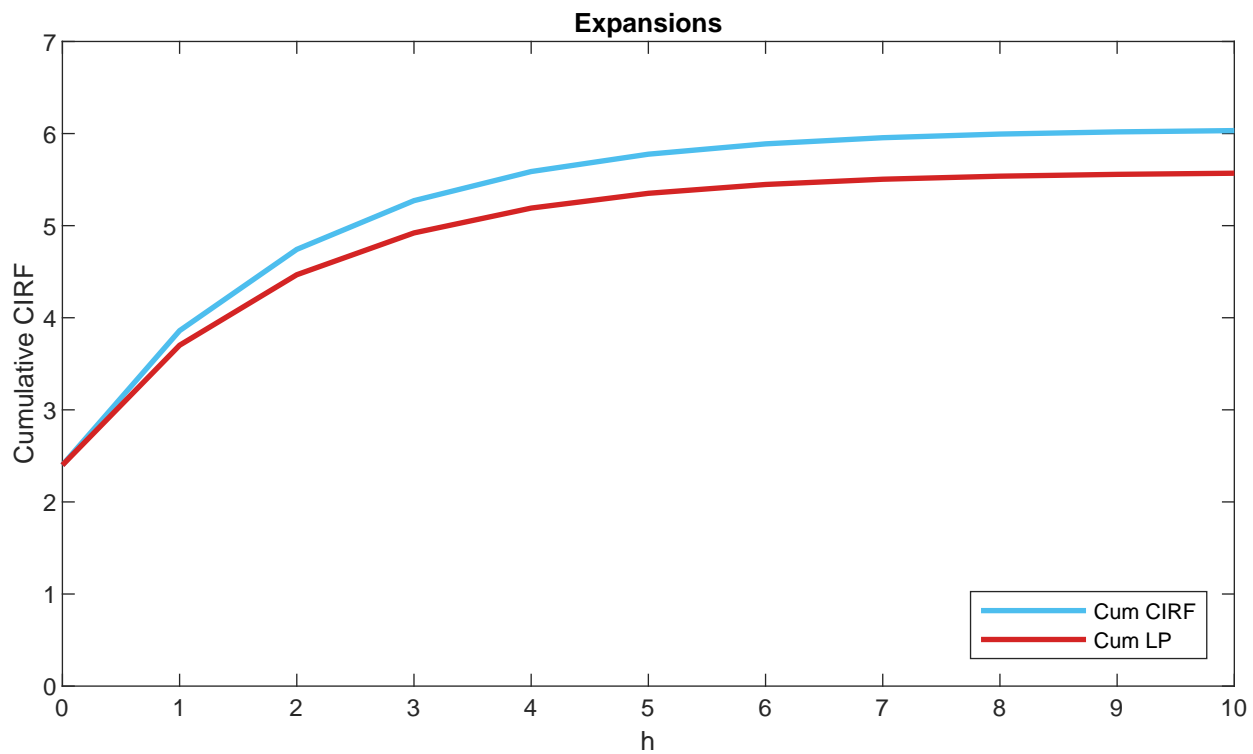


Figure 4: DGP2: Endogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Cumulative Effects

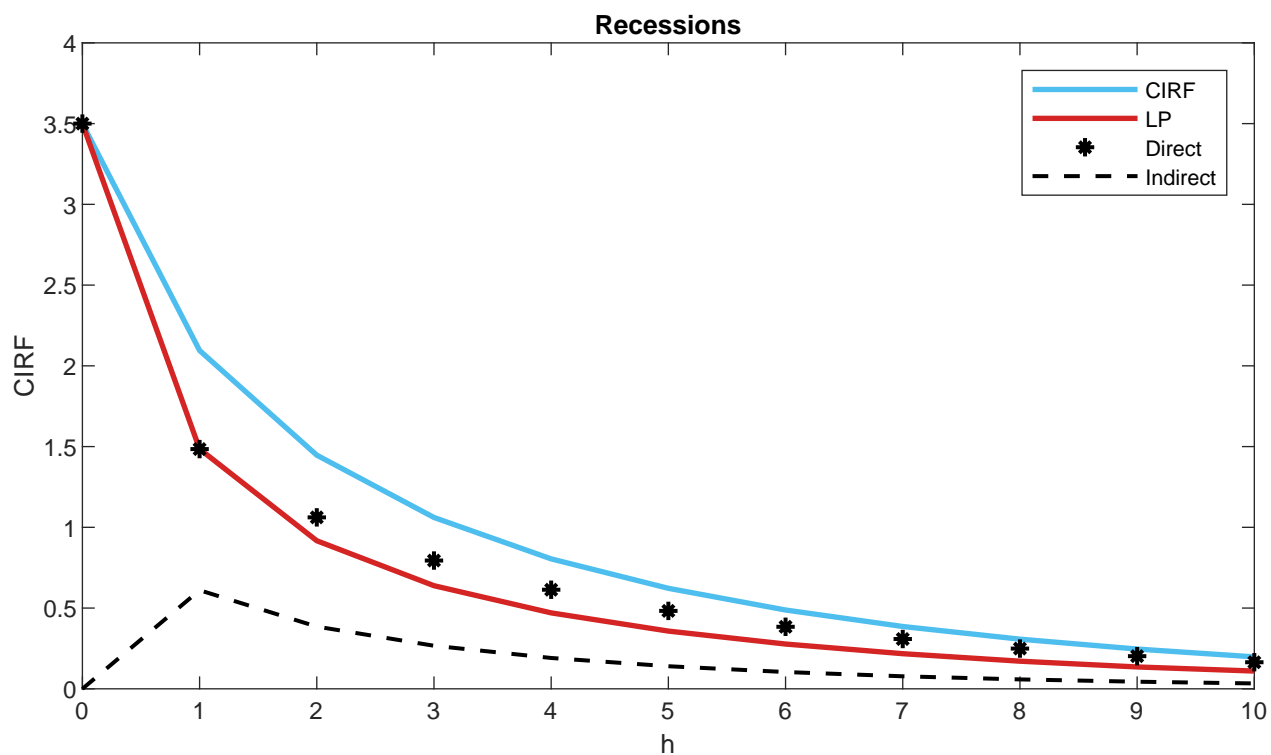
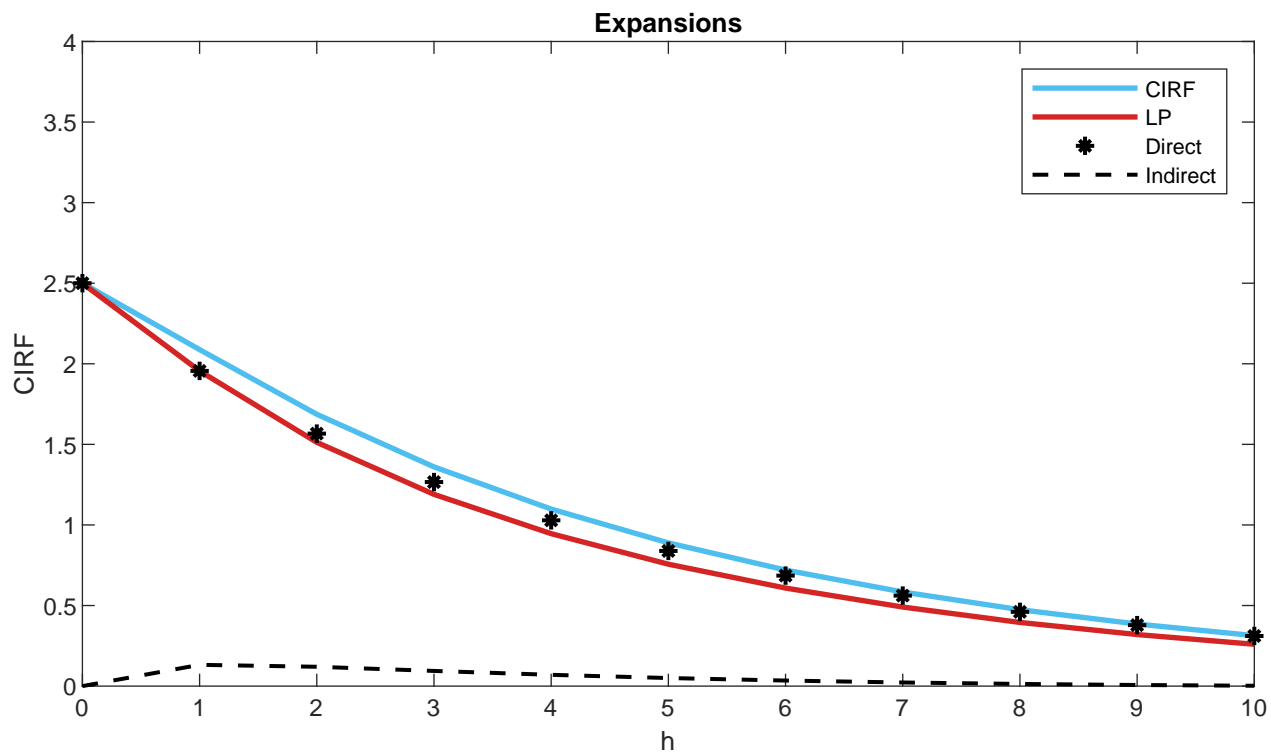


Figure 5: DGP3: Endogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Level Effects

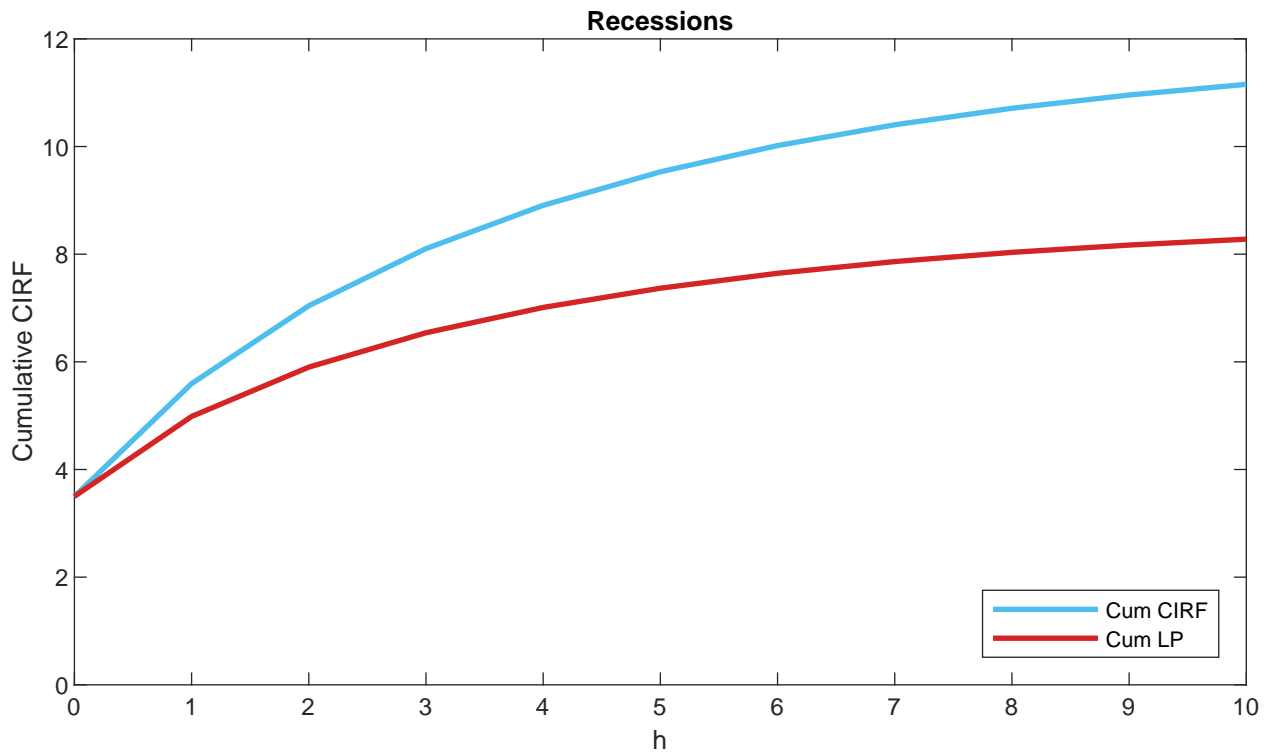
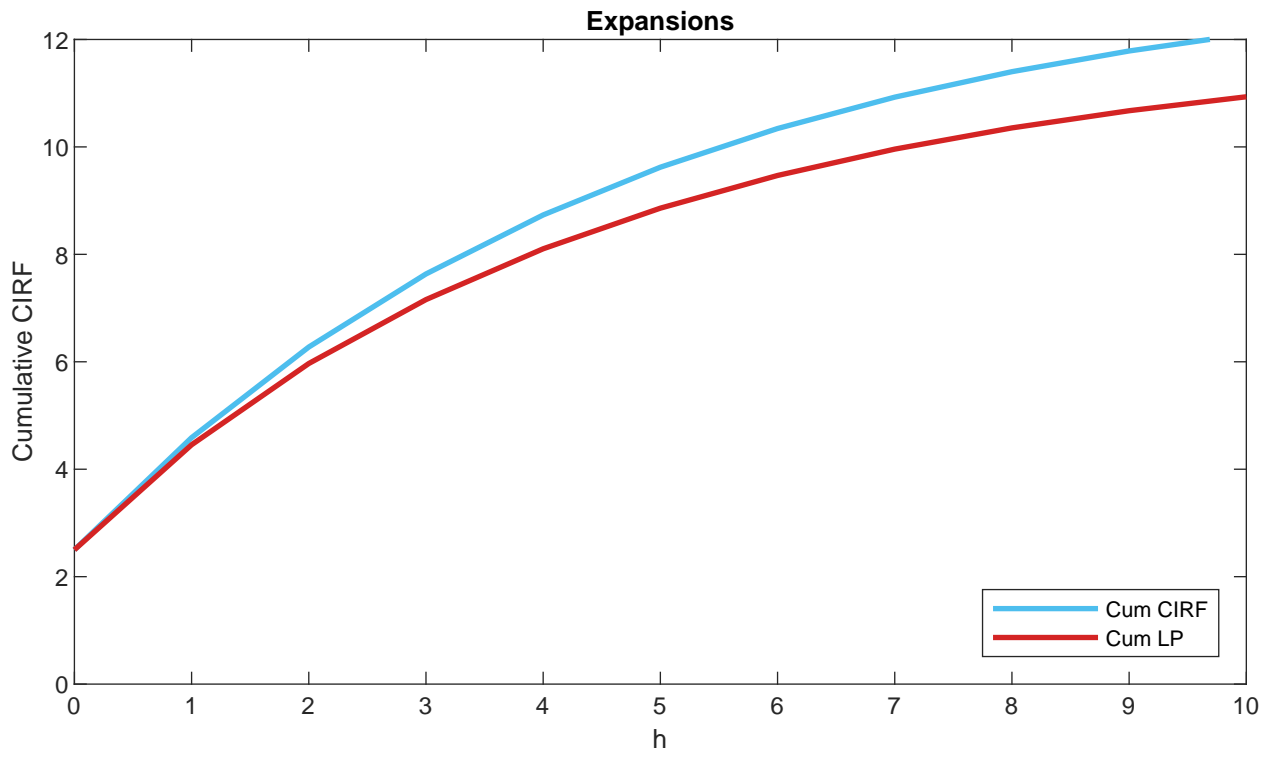


Figure 6: DGP3: Endogenous  $H_t$ ,  $x_t = \varepsilon_{1t}$ , Cumulative Effects

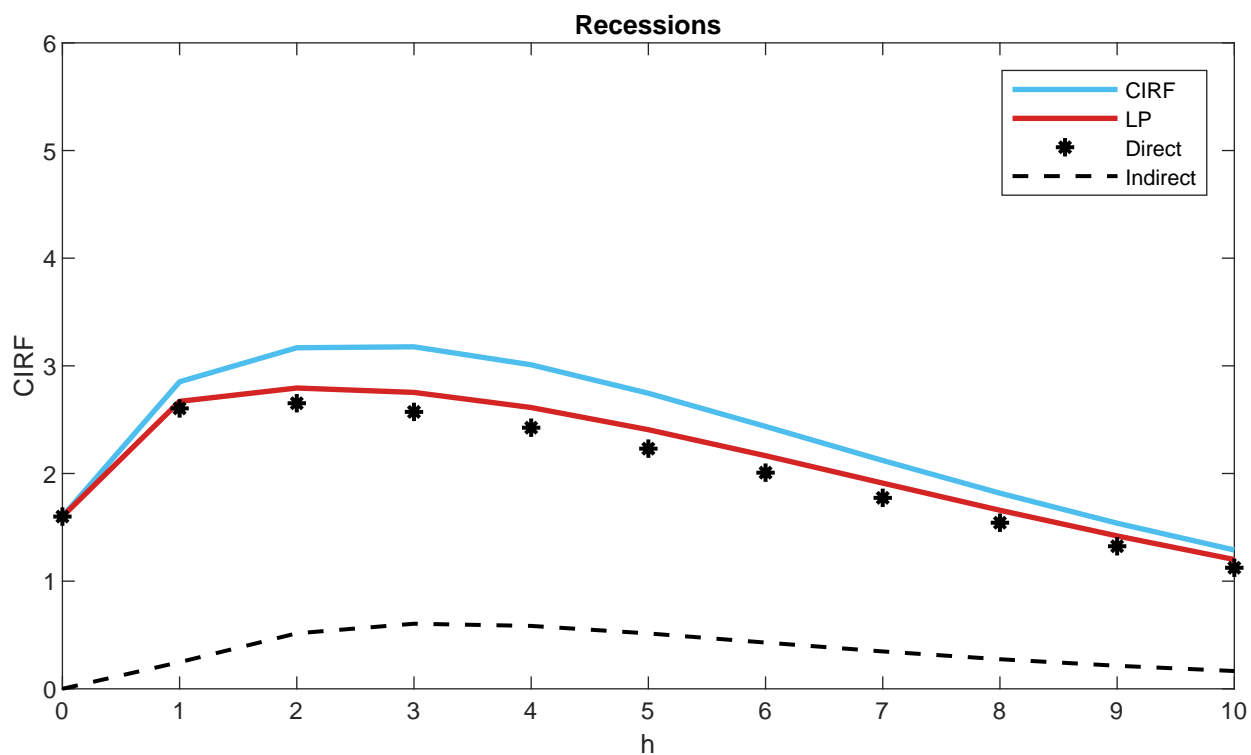
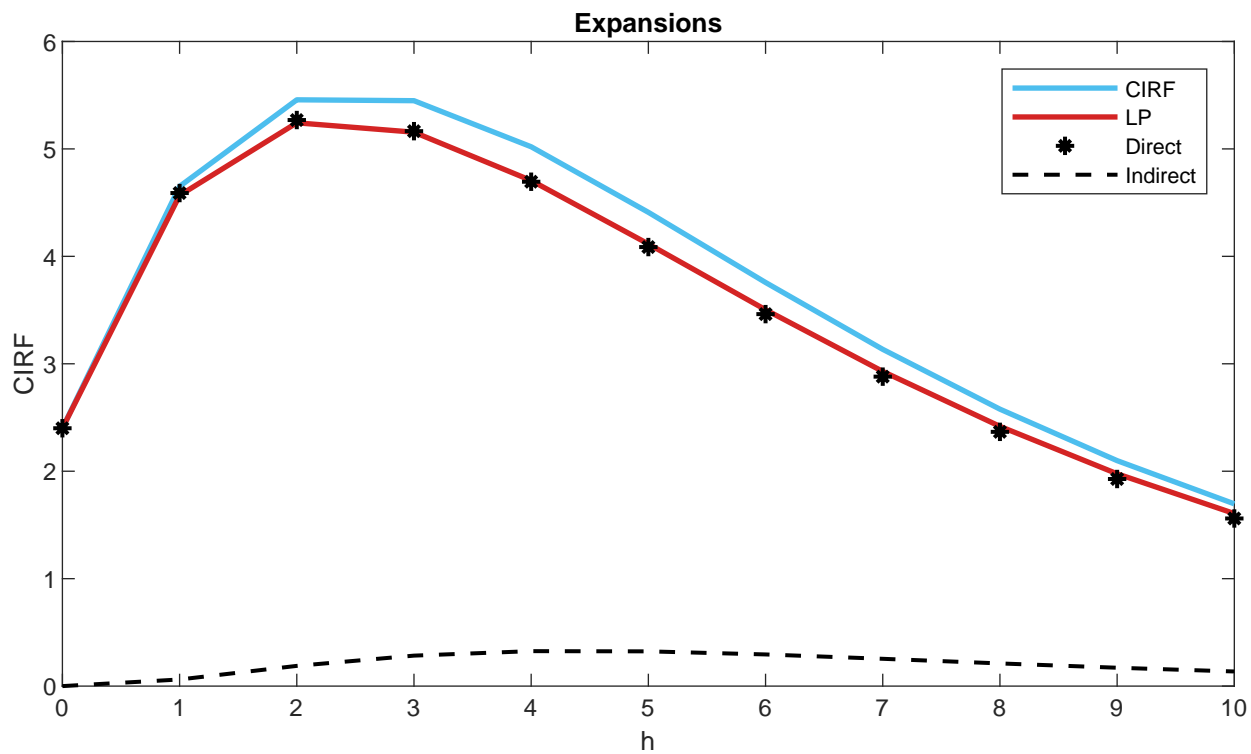


Figure 7: DGP4: Endogenous  $H_t$ ,  $x_t = 0.8x_{t-1} + \varepsilon_{1t}$ , Level Effects

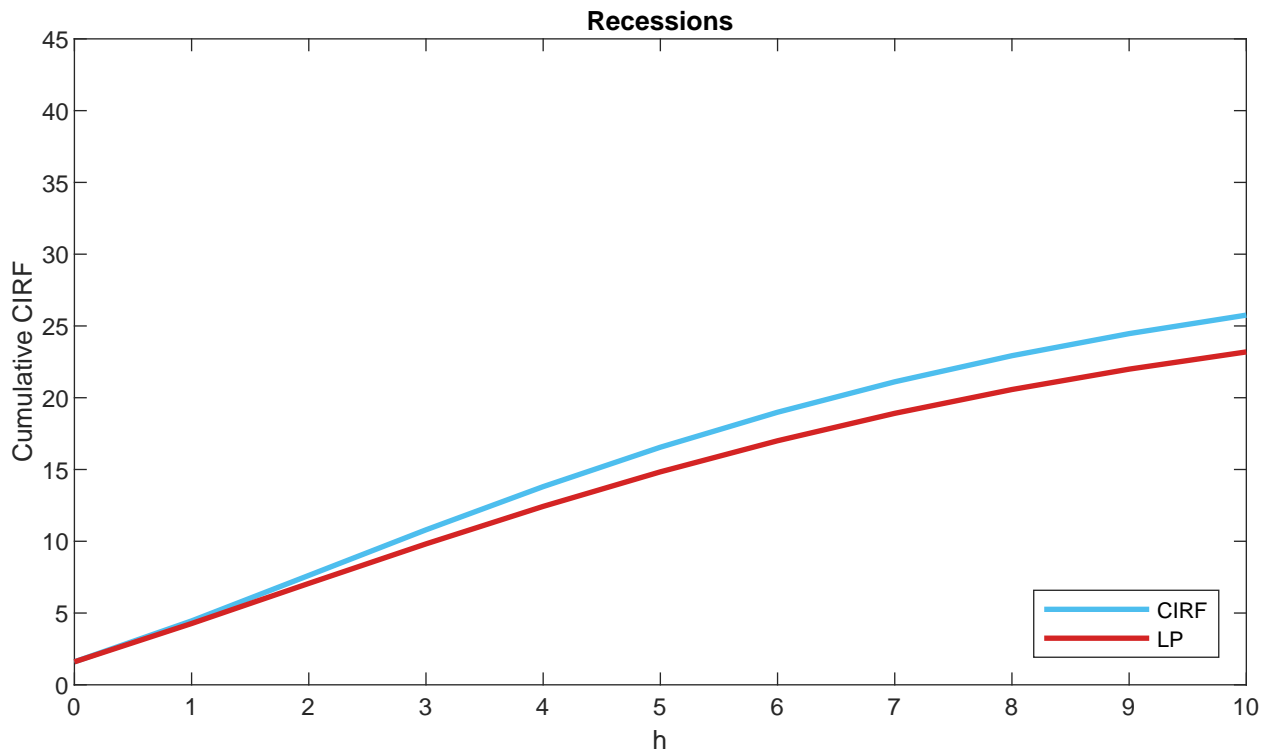
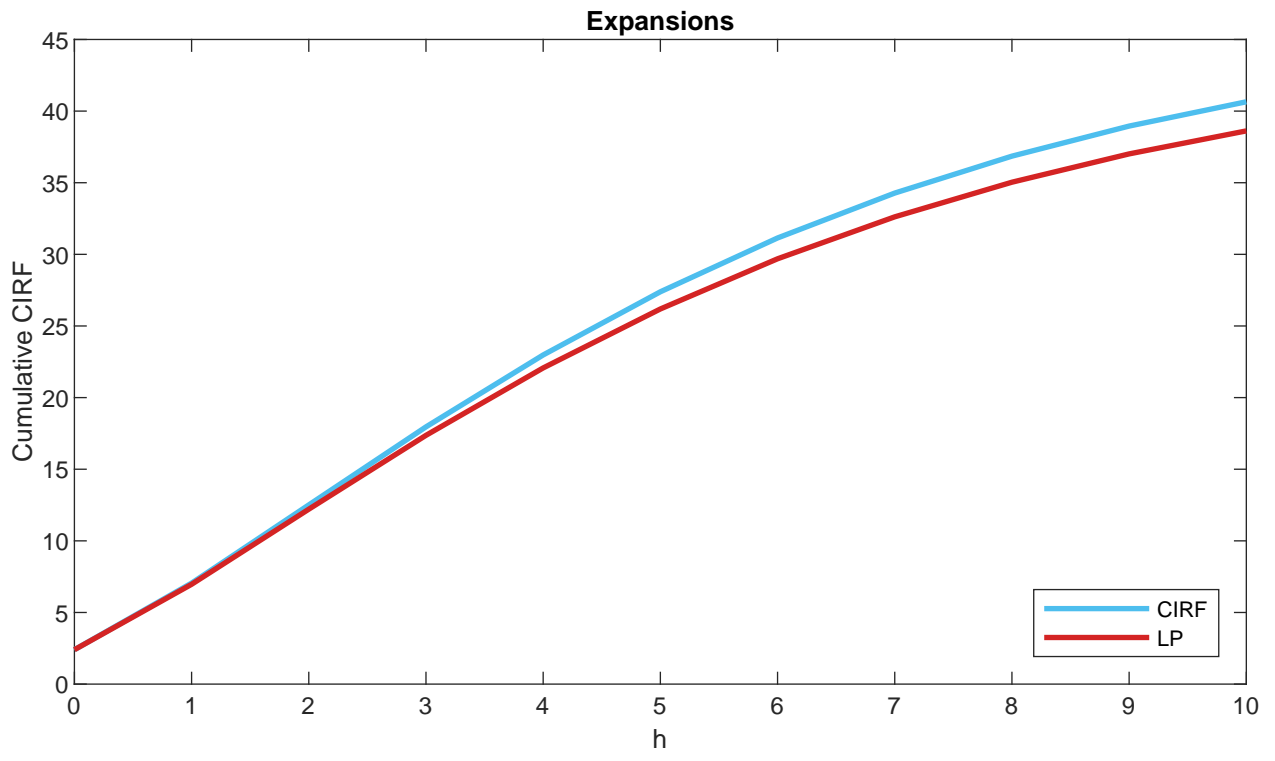


Figure 8: DGP4: Endogenous  $H_t$ ,  $x_t = 0.8x_{t-1} + \varepsilon_{1t}$ , Cumulative Effects

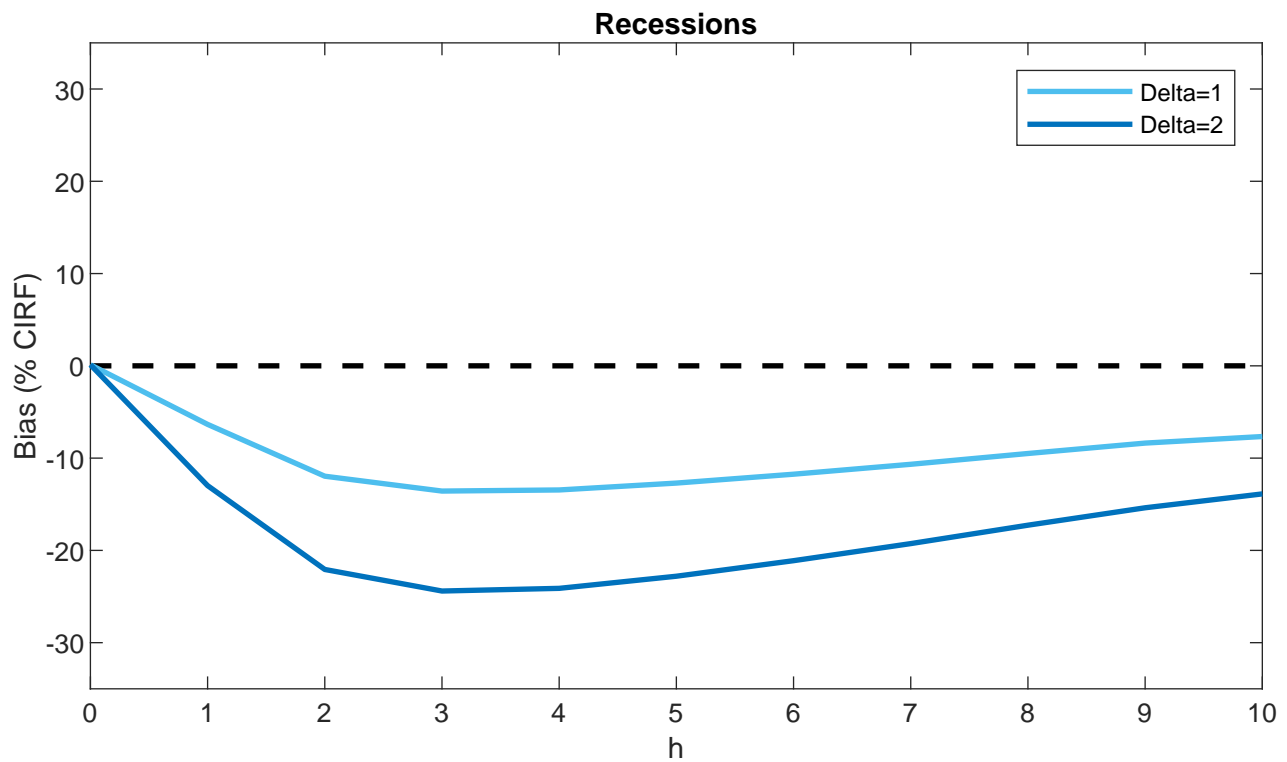
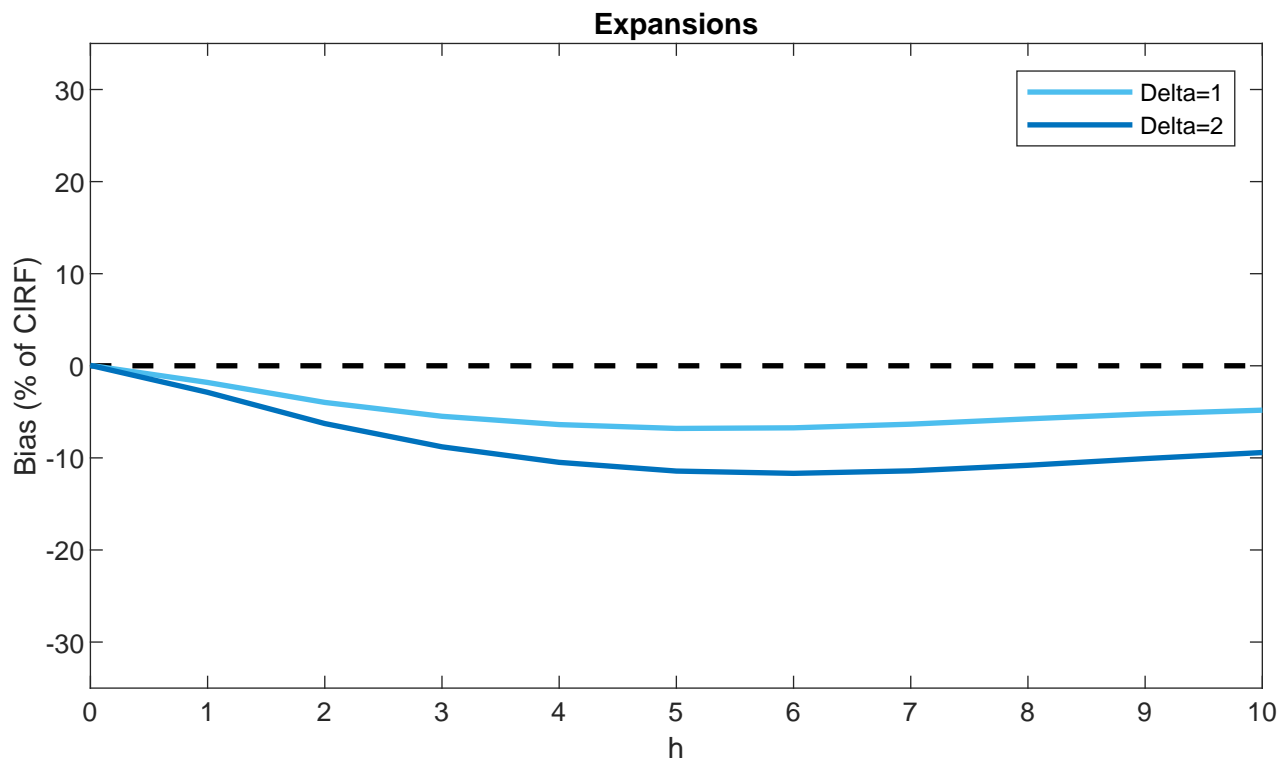


Figure 9: DGP4: Endogenous  $H_t$ ,  $x_t = 0.8x_{t-1} + \varepsilon_{1t}$ , Asymptotic Bias of the Level Effects