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Collusion in Repeated Auctions with Costless Communication

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Abstract

In this paper, we present a model of repeated first-price private value auctions in which the bidders have access to a cheap talk communication mechanism. In this framework, messages allow bidders to transmit their preference rankings over the goods to be auctioned, similar to Pesendorfer (2000). We show that collusion through this static mechanism not only dominates the static bid rotation mechanism presented by McAfee and McMillan (1992), but it is also not strictly dominated by the dynamic bid rotation mechanism presented by Aoyagi (2003). However, we show that asymptotic efficiency of collusion through increasing the number of ordered goods, presented by Pesendorfer (2000), demands patience rates to asymptotically approach one, making collusion increasingly more difficult to sustain. Finally, we study mechanisms through which the auctioneer may try to break bidders' collusion.

Keywords: Collusion, Auctions, Cheap Talk Communication, Repeated Games
JEL Codes: D44, C72, L41

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1 Introduction

The regular use of auctions to award government contracts to provide construction services (Kawai and Nakabayashi (2022), Porter and Zona (1993), and Conley and Decarolis (2016)), school supplies (Porter and Zona (1999) and Pesendorfer (2000)), and intermediate goods (Athey et al. (2011) and Schurter (2023)) may generate the perfect environment for collusion to emerge. Auctions occur on a regular basis, and the same limited number of companies regularly bid in these auctions. In this environment, while coordination among bidders is outlawed, simple messages that partially reveal preferences may not only be hard to detect, but may also be enough to allow for collusion.

In this paper, we present a model of repeated auctions in which bidders may sustain collusion by taking advantage of a simple cheap talk communication mechanism. This mechanism, initially suggested by Pesendorfer (2000), allows players to send and receive messages about their preference ordering over the goods to be auctioned in the stage game. We show that this simple communication scheme may significantly increase the expected gains from collusion compared to the static bid rotation presented by McAfee and McMillan (1992). By taking into account the preference orderings, the collusion strategy increases the likelihood that the good will be awarded to the bidder with the highest valuation, improving expected gains and efficiency. In contrast, we show that our static mechanism is not strictly dominated by a more complex dynamic rotation mechanism based on public strategies, such as the ones presented by Aoyagi (2003) and Green and Porter (1984). In fact, depending on the parameters, our mechanism delivers higher expected gains of collusion. Hence, there is a trade-off between information content and inefficient punishments. A dynamic rotation mechanism allows bidders to truthfully communicate their true valuation and consequently award the good to the bidder with the highest valuation. However, in order to obtain incentive compatibility, this dynamic mechanisms must rely on inefficient punishment on the equilibrium path. In contrast, our static rotation mechanism with ranking messages discloses significantly less information. Not only may rankings be identical across bidders, but also a lower ranking does not necessarily reflect a higher valuation. As a result, efficiency is lower. On the other hand, truth-telling is naturally incentive-compatible in this communication mechanism, so there is no need for inefficient punishment.

Another contribution of our model is to properly account for the incentives to deviate from collusion. Previous work by McAfee and McMillan (1992) and Pesendorfer (2000) takes compliance with the collusive strategy as given or unmodelled – what is usually known as *"side contract theory."* This assumption is important. We show that a result presented by Pesendorfer (2000) about the asymptotic efficiency of the ranking communication mechanism as the number of goods ranked increases relies heavily on the *"side contract theory."* By modelling the incentives to deviate explicitly, we show that patience rates may also asymptotically approach 1 as the ranking increase. Consequently, collusion becomes increasingly more difficult to sustain.

Finally, we discuss ways in which the auctioneer may attempt to break the bidders' collusion in this environment. As usual, increasing the number of potential bidders as well as making the schedule more concentrated may reduce the incentives for collusion. However, these options may be out of the auctioneer's hands, due to the need for specialized services, which lead to few potential bidders, and the regular need for the service or good, thereby inducing a repeated environment. In contrast, changing tiebreaker rules seem mostly ineffective, since the collusion strategy can adjust for that. Increases in the reserve price are somewhat ambiguous. On one hand, it does reduce the auctioneer's losses due to collusion. On the other hand, its impact on bidders' incentives to deviate from collusion is ambiguous.

Section 2 presents a brief literature review. Section 3 introduces the basic collusion model. Section 4 compares our mechanism gains to those of McAfee and McMillan (1992) and the ambiguity between the gains of the static bid rotation with ranking communication presented here and the dynamic scheme proposed by Aoyagi (2003). Section 5 evaluates the tools available to the auctioneer to either break collusive behavior or reduce the negative impact of collusion on auction revenues. Finally, Section 6 concludes the paper, summarizing the results. All proofs are in the Appendix.

2 Literature Overview

Collusion in first-price, private, and independent value auctions was first studied by McAfee and McMillan (1992). They presented a static model of collusion, i.e., a scheme in which the collusion rule remains fixed throughout the entire cooperation phase. In this model, punishment is not modeled, considering that collusion is based on the winner's identity and her bid, information considered public in the model. The authors argue that punishment must happen through repeated games. This approach is known as *"side contract theory."*

When choosing the collusion mechanism, given the multiplicity of equilibria that arise due to the need for a punishment scheme through repeated games, the authors focus on the mechanism that would maximize the *ex ante* joint profit of cartel participants. Furthermore, they introduce the question of a cartel's efficiency. A cartel would be efficient if it designated the player with the highest value as the winner of the auction, as long as her valuation was higher than the reserve price. Obviously, efficiency would imply a greater joint profit, so a profit-maximizing cartel would prefer to obtain an efficient result, as long as it had incentive compatibility. In terms of mechanisms available for collusion, McAfee and McMillan (1992) consider the possibility of the formation of cartels in two environments:

- Weak Cartel: There is no possibility of making payments between participants;
- Strong Cartel: The cartel has the possibility of making payments between participants.

McAfee and McMillan (1992) then look at the optimal collusion mechanisms in each case, considering that the auctioneer takes a passive stance in the auction, simply announcing a reserve price and selling the good to the participant who makes the highest offer. In the event of a tie, the auctioneer evenly draws the good among the players who made the highest bids.

In the case of weak cartels, the mechanism that would maximize the joint ex ante revenue of the cartel, among all possible mechanisms – with or without correlation between bids – would be one in which every player with a valuation above the reserve price would make an offer equal to the reserve price, leaving the auctioneer to draw the winner. Obviously, this rule does not generate an efficient result. However, the authors show that there is no way to obtain a rule of efficient collusion for the case of weak cartels.

In the case of strong cartels, the optimal mechanism would ask each player to report its valuation to the cartel. The player with the highest valuation would be designated the winner of the auction, paying the auctioneer the reserve price and making cash transfers to other players for a total amount equal to the difference between the expected value of the second largest valuation and the reserve price. Obviously, this mechanism is equivalent to carrying out a first-price pre-auction between collusion participants before the official auction. Note that in this case the cartel would be efficient.

Based on this result, the authors address the question of the auctioneer's behavior. Regarding the reserve price, since the optimal reserve price in the case of collusion is higher than the competitive reserve price, detection of collusion by participants leads to raising the reserve price charged by the auctioneer as a way of mitigating the effects of the cartel. According to McAfee and McMillan (1992), when there are a small number of participants at auction, the gains from collusion in an environment in which the reserve price has been raised will be smaller than the joint gains in a competitive environment with a low reserve price. Therefore, if the discount rate is low enough, an auction with few participants would not generate incentives for collusion, since the gains in the short term will be smaller than the losses in the future, when the auctioneer raises the reserve price to combat collusion. This result is presented for the case in which player valuations are taken from a distribution $U[0, 1]$.

A key criticism of McAfee and McMillan (1992) is their reliance on *"side contract theory."* A model that aims to internalize the issue of punishment is presented by Johnson and Robert (1999). This model presents a static weak collusion model in first-price, private, and independent value auctions, repeated infinitely, in which each player knows her valuation before each auction takes place. From this environment, the authors create a collusion model based on Abreu et al.'s (1990) methodology and the idea presented by Green and Porter (1984) that there is the possibility of stochastic shocks, in this case, the possibility of a player with a high enough valuation to accept any future punishment. This behavior is not taken into account by the scheme of equal offers presented by McAfee and McMillan (1992), and, according to Johnson and Robert (1999), is an intrinsic feature of modeling repeated games. Thus, the collusion strategy in Johnson and Robert's (1999) model internalizes the possibility of players deviating from collusion whenever it is profitable, allowing them to deviate and be punished for a certain number of rounds. As a result, the optimal offering scheme now has a ladder format, with sections of collusion and others of competition in the offers.

Regarding the instruments available to the auctioneer, Johnson and Robert (1999) show that the impact of the reserve price becomes ambiguous. They then discuss other instruments at the auctioneer's disposal, in particular, the use of ceilings for possible offers and the use of tiebreaker rules that indicate the winner ex ante. In the case of the offer ceilings, there would be a stimulus to break the collusion by increasing the players' earnings in the case of competition, while the ex ante rule would hinder the collusion scheme, increasing the minimum patience rate required to maintain the collusion.

However, the possibility that higher valued players will deviate is reduced by the introduction of mechanisms that allow the player with the highest valuation to win the auction, rescuing the efficiency of the collusion scheme while avoiding problems of incentive compatibility. One way to achieve this is to treat a scheme of dynamic collusion, in which the collusion rule depends on past history. For example, Skrzypacz and Hopenhayn (2004) present a model in which the current stage auction winner has a lower continuation payoff. Basically, this mechanism attempts to imitate, through transfers of future expected payoff, side payments among bidders such as in a strong collusion scheme, in order to rescue efficiency, which is obtained in asymptotic terms as the number of cartel participants grows.

In Johnson and Robert (1999) and Skrzypacz and Hopenhayn (2004), payments made to the auctioneer may be higher than the reserve price. Aoyagi (2003) considers the case in which the winning bid equals the minimum payment. In order to sustain such a result, he considers a model with two potential bidders in which each bidder communicates her valuation to a *center* that coordinates the bidding. This center then selects the player with the highest valuation; so she wins the auction while paying the reserve price. To avoid incentive-compatibility issues, the higher the reported valuation, the greater the probability of initiating a punishment phase of m periods, in which the player being punished wins the auction only if the other player has the valuation below the reserve price. Obviously, this scheme generates inefficiency, although it generates gains compared to the tacit static collusion scheme presented by McAfee and McMillan (1992).

Finally, regarding communication mechanisms in collusion in first-price, private, and independent auctions, Pesendorfer (2000) presents a model in which agents send messages regarding their preference ordering. He shows that, by increasing the number of goods ordered, the collusion scheme becomes asymptotic efficient. An important caveat is that, similarly to the case of McAfee and McMillan (1992), Pesendorfer (2000) also relies on *"side contract theory;"* that is, he does not model the incentive to deviate from the collusion strategy.

3 Model

In this section, we present a collusion model in a repeated first-price auction with a private valuations' environment in which bidders may send costless messages to each other, following Pesendorfer (2000). These messages contain information about the ordering of the bidders' preferences as well as the bidders' intent to participate in the auction. We show that the introduction of this cheap-talk message mechanism allows players to significantly improve the cartel's revenues compared to the pure rotation of offers, as presented by McAfee and McMillan (1992). Furthermore, we show that the dynamic offer rotation mechanism proposed by Aoyagi (2003) does not necessarily generate a higher expected revenue than the static mechanism with cheap-talk messages presented in this paper. We also present comparative statics, showing how variations in the distribution of private values $F(\cdot)$ may impact the bidders' incentives to remain in the cartel.

We then extend the analysis in Pesendorfer (2000) by considering his result of asymptotic efficiency through the increase in the number of goods ordered by the communication system. We show that, taking into account the repeated game structure and the need for an incentive-compatible punishment phase, the efficiency result can only be obtained through an increasingly higher patience rate.

Finally, we discuss the cartel's incentive compatibility through the one-shot deviation principle for the case in which the valuations are drawn from a distribution $U[0, 1]$.

Costless communication mechanisms (cheap talk) are known for generating multiple equilibria, which is clearly a drawback in our analysis. However, our goal here is to provide an intuitive mechanism with clear implications for the literature on collusion in repeated auctions. Toward this goal, we also impose the rich-language assumption, in which we impose the condition that players are not easily misunderstood.

3.1 Environment

We consider an environment with repeated auctions in which bidders and auctioneer discount future periods by $\delta \in (0, 1)$.[1] The supergame is divided into stage games, with each stage game having m periods, that is, m auctions.

There are n potential bidders. Bidders are risk-neutral, infinitely lived, and *ex ante* identical. A bidder's valuation of a good to be auctioned is a private value θ, which is a random draw from a continuously differentiable distribution $F(\cdot)$ with support $[0, 1]$. For simplicity, we assume that $F(\cdot)$ is common knowledge and the same for all bidders. There is no cost to making a bid. Finally, there is a reserve price $b_0 \geq 0$ set by the auctioneer that is constant and known by all participants.

At the beginning of each stage game, before the first auction takes place, bidders know their valuations for the stage game auctions and send a signal about their valuations to the other players, receiving their signals simultaneously. Based on her own valuation and the signals received, each participant chooses her bid for the first auction. At the end of the first period, players know the auction's winner and winning bid. For each follow-up period in the stage game, the player decides her bid based on her valuation, the signals received from the other players, and the winning bids from the previous periods' auctions within the stage game.

[1]We can also interpret this environment as a finite game whose probability of a new round is always positive (so we have $0 < \delta < 1$).

Colluding Strategy: From the stage game structure, we consider the following collusion strategy:

First Phase – Communication: Each player sends two messages to the opponent:

- The ordering of preferences for the next goods that will be auctioned.

  This ordering is given by the following function: $pos(\theta_i^k) : [0,1]^m \to \{1, ..., m\}$, which indicates the position of good k in the ordering of player i among the goods auctioned in the stage game. Lower ranking indicates a preferred good. Note that, unlike in the case of valuation, there is a significant possibility of a tie between bidders' rankings. Consider, for example, the case of two goods to be auctioned: k and j.[2] In this case, we have the following possible ties:

  a. $pos\left(\theta_i^k\right) = pos\left(\theta_{-i}^k\right) = 1$ and $pos\left(\theta_i^j\right) = pos\left(\theta_{-i}^j\right) = 2$ or;
  b. $pos\left(\theta_i^j\right) = pos\left(\theta_{-i}^j\right) = 1$ and $pos\left(\theta_i^k\right) = pos\left(\theta_{-i}^k\right) = 2$.

  where θ_i^k is bidder i's valuation for the good k.

- A signal that indicates whether her valuation is above or below the reserve price set by the auctioneer.

Second Phase – Auctions: Cartel participants bid according to the following strategy:

a) Consider good k's auction: Suppose that at least two participants have valuations above b_0. If $pos(\theta_i^k) < pos(\theta_{-i}^k), \forall -i$, participant i must bid $b = b_0$, while all opponents must bid $\bar{b} < b_0$. In contrast, if $pos(\theta_i^k) = pos(\theta_j^k) < pos\left(\theta_{-(i,j)}^k\right)$, i.e., the ranking for good k is the same for bidders i and j and lower than the one for all other participants, bidders i and j should bid $b = b_0$ while all other cartel participants should bid $\bar{b} < b_0$. As a result, the auctioneer chooses the winner at random between participants i and j.

b) In case any player deviates, all bidders switch to competition from the next period on.

We now show that this strategy has a communication system that reveals the truth and produces gains relative to competition. We focus on the case $n = m = 2$.

Proposition 1 *In the case of two bidders and two auctions per stage game, it's optimal for each bidder to truthfully report her preference ordering and if valuations are above or below the reservation price.*

Pesendorfer (2000) shows that in the case of two bidders and two auctions per stage game, the equilibrium of the communication game is unique. In the case where there are more than two players or more than two auctioned goods per period, we can have multiple equilibria. However, we focus on the case in which all players tell the truth.

[2] The use of k and j here is a bit of abuse of notation. In the proofs, we focus on k and $k - (-1)^k$, because we have to compare the good sold in the odd period with the good sold in the even subsequent period.

We now consider the gains from collusion. First, let's consider bidder i's expected profits. Note that, in addition to winning the auction whenever the opponents' valuations are less than b_0, while her valuation is greater than b_0, player i wins good k's auction if:

1. bidder i's ranking of good k is less than other bidders' ranking of good k, while her valuation is greater than b_0: agent i wins the auction by paying the reservation price and her expected gain is $E\left[\theta_i^k - b_0\right]$.

2. bidder i's ranking of good k is equal to some other $j-1$ agents' ranking of good k – being lower than the ranking of the others – and their valuation is higher than b_0: agent i wins the auction with probability $\frac{1}{j}$, where $j-1$ is the number of agents who tied with i in the ranking of good k, with the expected gain being $\frac{1}{j}E\left[\theta_i^k - b_0\right]$.

As a result, the bidder's expected gains from collusion in the case with two bidders and two auctions per period (calculations presented in Appendix A):

$$\left[\frac{1+2F\left(b_0\right)+F\left(b_0\right)^2}{4}\right]\int_{b_0}^{1}\left(\theta-b_0\right)f\left(\theta\right)d\theta+\frac{1-F\left(b_0\right)}{2}\int_{b_0}^{1}\left(\theta-b_0\right)F\left(\theta\right)f\left(\theta\right)d\theta.$$

in the case in which $b_0 = 0$, the expression simplifies to:

$$\sum_{k=0}^{\infty}\delta^k\left\{\frac{1}{2}E\left(\theta F\left(\theta\right)\right)+\frac{1}{4}E\left(\theta\right)\right\}. \tag{1}$$

From expression (1), let us see the conditions for which the collusion gains are greater than competition. The *ex ante* gain from collusion is greater than competition if:

$$\sum_{k=0}^{\infty}\delta^k\left\{\frac{1}{2}E\left(\theta F\left(\theta\right)\right)+\frac{1}{4}E\left(\theta\right)\right\}\geq\sum_{k=0}^{\infty}\delta^k E_\theta\left[\int_0^\theta F(X)dX\right]. \tag{2}$$

where the right-hand side of (2) is the bidder's expected payoff in competition. Making the appropriate calculations (see Appendix A), we obtain the following expression:

$$E\left(\theta F\left(\theta\right)\right)\leq\frac{5}{6}E\left(\theta\right). \tag{3}$$

This result indicates that, in order for collusion to be profitable, the private value distribution cannot be concentrated close to 0. The intuition for this result comes from the fact that, when valuations are low and close to each other, gains from collusion in reducing the price paid by the winning bid are small. In contrast, the loss of collusion efficiency remains because the player faces the possibility of losing the asset even though she has the highest valuation, while competition generates an efficient result.

However, the collusion strategy must not only satisfy the participation constraint but must also be incentive-compatible in order to avoid deviations. Therefore, we must show that the collusion strategy is a perfect subgame equilibrium of the supergame. According to Fudenberg and Tirole (1991),[3] this can be guaranteed if we show that no player has an incentive to deviate at an arbitrary stage of the game (known as the one-shot deviation principle). Let us consider the case of the player with the greatest incentive to deviate in the stage game, i.e., the player who loses the auction in the first period and wins the auction in the second period. In this case, the one-shot deviation is:

$$0 + \delta(\theta_i^1) + \frac{1}{2}\frac{\delta^2}{1-\delta}\left[E\left(\theta F(\theta)\right) + \frac{1}{2}E(\theta)\right] \geq (\theta_i^0 - \varepsilon) + \frac{\delta}{1-\delta}\left[\int_0^1 F(\theta)d\theta - \int_0^1 F(\theta)^2 d\theta\right] \quad (4)$$

Since ε can be arbitrarily small, we take $\varepsilon \to 0$ and omit this parameter. Consider, at first, that $\delta(\theta_i^1) = \theta_i^0$ (i.e., consider that the player has the same gain by waiting and winning next period's auction or deviating this period). Hence, we have:

$$\frac{\delta^2}{1-\delta}\left[\frac{1}{4}\left(1 + E\left[\theta\right] - \int_0^1 F(\theta)^2 d\theta\right)\right] \geq \frac{\delta}{1-\delta}\left(1 - E\left[\theta\right] - \int_0^1 F(\theta)^2 d\theta\right) \quad (5)$$

simplifying it:

$$\delta \geq \frac{4\left(1 - E[\theta] - \int_0^1 F(\theta)^2 d\theta\right)}{\left(1 - \int_0^1 F(\theta)^2 d\theta + E(\theta)\right)} \quad (6)$$

Define $\bar{\delta}$ as the minimum value of δ that can sustain collusion. We are able to show the following result:

Proposition 2 *Consider two distribution functions F_1 and F_2. If $F_1(\theta)$ stochastically dominates $F_2(\theta)$ in second order, we have that $\bar{\delta}_1 \leq \bar{\delta}_2$, where δ_i is the discount rate relative to F_i.*

The result above gives an intuition about how the incentive to deviate from the colluding strategy changes as the variance of the private valuations decreases. The smaller the variance – i.e., the smaller the uncertainty regarding my and my opponents' valuation of the goods to be auctioned in the future – the easier it is to maintain collusion.

Notice that the simplifying assumption that $\delta(\theta_i^1) = \theta_i^0$ is not fundamental to Proposition 2, as we show in Lemma A.2 in the Appendix.

4 Comparison to the Literature

We now compare our results about collusion mechanisms to those previously presented in the literature. First, we show that the introduction of the communication mechanism generates gains compared to the static bid rotation strategy presented by McAfee and McMillan (1992).

Proposition 3 *Consider the introduction of a cheap-talk communication mechanism that allows bidders to report their ordering of preferences to each other. In the case of two players and two*

[3] Theorem 4.2, page 110.

goods, adding such communication mechanism to a static bid rotation strategy leads to revenue gains over a strategy in which all players submit identical bids.

We now compare the expected profits from our strategy to the dynamic bid rotation presented by Aoyagi (2003). Toward this goal, we present some definitions based on the work by Aoyagi (2003). First, consider the case of an efficient collusion scheme, ignoring issues of incentive compatibility. In this case, the expected profit per contract in the two-player case is:

$$g^* = \int_{b_0}^{1} (\theta - b_0) F(\theta) f(\theta) \, d\theta.$$

Define $\overline{g}$ the payoff of the bidder who wins the auction whenever her valuation is above b_0, i.e.:

$$\bar{g} = \frac{1}{1 - F(b_0)} \int_{b_0}^{1} (\theta - b_0) f(\theta) \, d\theta.$$

Similarly, let's define $\underline{g}$ as the payoff of the bidder who only wins when her opponent has a payoff below the reservation price b_0:

$$\underline{g} = \frac{F(b_0)}{1 - F(b_0)} \int_{b_0}^{1} (\theta - b_0) f(\theta) \, d\theta.$$

Since the optimal aggregate profit is given by $2g^*$, Aoyagi (2003) shows that:

$$2g^* > \bar{g} + \underline{g}.$$

We now present a result showing that the dynamic bid rotation proposed by Aoyagi (2003) does not strictly dominate the static bid rotation with communication presented in the current paper. Furthermore, we present an example in which Aoyagi's (2003) collusion strategy generates a strictly lower joint profit than our strategy.

Proposition 4 *The dynamic bid rotation scheme proposed by Aoyagi (2003) does not strictly dominate the static bid rotation mechanism with communication.*

The reason why a static bid rotation with ranking communication may be superior to Aoyagi's (2003) in certain cases is that the cost of the punishment phase needed to induce bidders to communicate their true valuations may outweigh the benefit. The introduction of a communication mechanism that, while revealing less information, does not require additional punishments may be superior, as long as sufficient information is revealed. A good way to show this inability to rank these strategies is to present an example in which the collusion scheme we present may be superior to the dynamic bid rotation.

Example 1 *Consider the case in which the valuations are taken from a uniform distribution* $U[0,1]$. *Assume four goods are being auctioned per period* $(m = 4)$, *and there are two bidders* $(n = 2)$ *and a* *zero reserve price* ($b_0 = 0$). *In this case, comparing the efficient collusion scheme's expected profit against the collusive strategy with a ranking communication system, we have:*

$$\frac{\textit{Profit per Auction}}{g^*} = 0.9375.$$

in contrast, comparing the efficient collusion scheme's expected profit against the dynamic bid rotation in Aoyagi (2003), we have:

$$\frac{\textit{Aoyagi}}{g^*} = 0.8860.$$

4.1 Asymptotic efficiency and subgame perfection

We now consider the trade-off between asymptotic efficiency and subgame perfection. From Pesendorfer (2000), we know that, for $b_0 = 0$, $m \to \infty$ implies that *return* $\to g^*$. In other words, by increasing the number of stage auctions per period, the collusion strategy with ranking messages approaches asymptotically the efficient collusion scheme's expected profits. However, this result does not take into account incentive compatibility. When we study the incentives for players to deviate, we obtain the following proposition:

Proposition 5 *When $m \to \infty$, the worst-off player always has an incentive to break with the collusion rule.*

To highlight this problem, we consider the particular case in which private values are draws from a uniform distribution $U[0,1]$. In this example, we can clearly show the trade off between efficiency and the incentive to deviate from the collusion.

4.2 Uniform distribution case

Let's consider initially the case in which $m = n = 2$. Solving for the case in which the θ_0 follows a uniform distribution $U[0,1]$ we obtain the following expression for the ex ante expected profit when the reservation price is b_0:

$$\frac{1}{1-\delta}\left\{\frac{7 - 10b_0 + 2b_0^3 + b_0^4}{24}\right\}.$$

assuming $b_0 = 0$, the expected profit simplifies to $\frac{1}{1-\delta}\left(\frac{7}{24}\right)$.

Let's address the bidders' incentive to deviate from collusion. In the case of the player who loses the auction at $t = 0$, there is no incentive to deviate if:

$$\delta(\theta_i^1 - b_0) + \frac{\delta^2}{1-\delta}\left\{\frac{7 - 10b_0 + 2b_0^3 + b_0^4}{24}\right\} \geq \left[\theta_i^0 - b_0\right] + \frac{\delta}{1-\delta}\left\{\frac{2b_0^3 - 3b_0^2 + 1}{6}\right\}. \tag{7}$$

Before we show some examples, let's consider some general results:

Lemma 1 *If $F(\theta)$ follows a uniform distribution $U[0,1]$, we have:*

- $\dfrac{\partial \bar{\delta}}{\partial \theta_i^0} > 0$;
- $\dfrac{\partial \bar{\delta}}{\partial \theta_i^1} < 0$.

Figure 1 shows how the variation of θ_i^1 and θ_i^0 affects $\bar{\delta}$ for 10,000 random draws from a $U[0,1]$ distribution. The format of the graph is mainly because $\theta_i^1 \geq \theta_i^0$, so that the graph accumulates close to the highest values of θ_i^1 while moving away from the highest values of θ_i^0 (given that, for $\theta_i^0 = 1$, necessarily $\theta_i^1 = 1$).

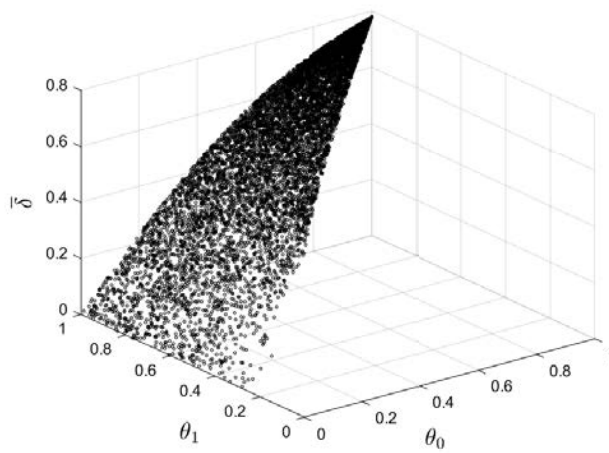

Figure 1: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods

This graph shows the minimum delta value that supports the collusion $(\bar{\delta})$, given the auction participants' valuations for the goods that will be sold in the stage game, θ_0 and θ_1, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current period (θ_0) and wins the auction of the following period, in an environment with zero reserve price $(b_0 = 0)$ and two goods per stage game $(m = 2)$.

In Figure 2, we show the relationship between $\Delta\theta = \theta_i^1 - \theta_i^0$ and $\bar{\delta}$, i.e., the difference between the valuation of the good that the player wins and the valuation of the good that the player gives up and the minimum patience rate necessary to maintain the collusion. As expected, the smaller $\Delta\theta$ (the greater the possible gain in the deviation), the greater the $\bar{\delta}$ required to maintain collusion.

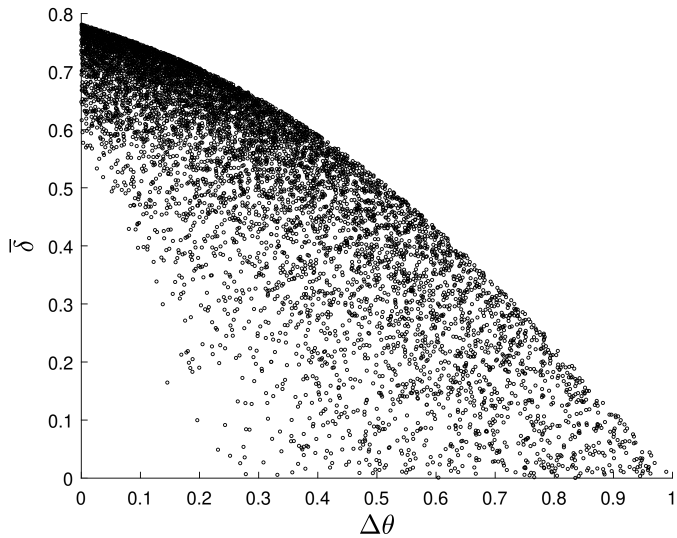

Figure 2: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods
This graph shows the minimum delta value that supports the collusion $(\bar{\delta})$, given the difference in the players' valuations of the good sold in the stage game, θ_0 and θ_1, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current period (θ_0) and wins the auction of the following period, in an environment with zero reserve price $(b_0 = 0)$ and two goods per stage game $(m = 2)$.

Let us now consider the case in which we have four goods auctioned in the stage game $(m = 4)$, two players $(n = 2)$ and zero reservation price $(b_0 = 0)$. In this case the expected profit is:

$$\frac{1}{1-\delta}\left[\frac{1}{8}\left\{6E\left(\theta F\left(\theta\right)\right) + E\left(\theta\right)\right\}\right].$$

For the uniform case, we observe that the expected profit is higher than in the previous case $\left(\frac{5}{16} > \frac{7}{24}\right)$.

Let's now consider the question of the incentive to break the collusion. Consider the case in which agents have the following orderings: $(\theta_1, \theta_2, \theta_3, \theta_4)$ and $(\theta_2, \theta_3, \theta_4, \theta_1)$, where θ_1 is the good sold in the fourth period and θ_2 is sold in the first period of the stage game. Therefore, in this case, the one-shot deviation principle gives us:

$$\delta^4\left(15 - 48\theta_1\right) + \delta^3 48\theta_1 + \delta\left(48\theta_2 - 8\right) - 48\theta_2 \geq 0.$$

Graphically, treating again the relationship between the difference in the valuation of the goods that the player gains if she remains in the collusion and the valuation of the goods that the player would gain if she deviates from collusion and the minimum patience rate to sustain the collusion, we obtain Figure 3. Obviously, we observe an increase in the minimum value necessary to sustain collusion $(\bar{\delta})$ for any difference between θ_1 and θ_2.

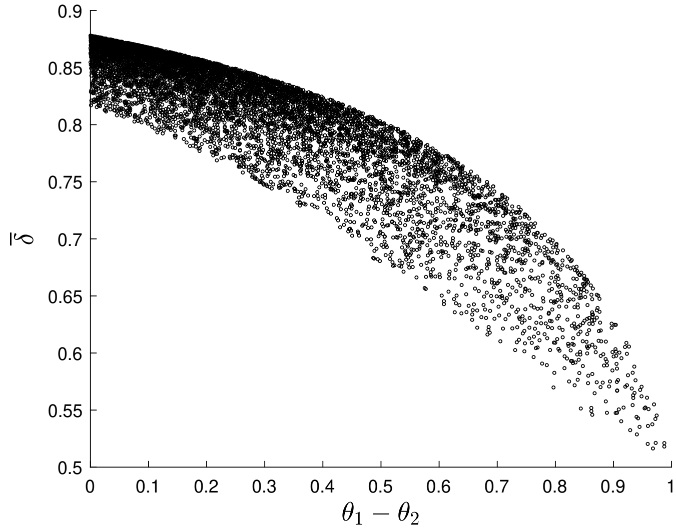

Figure 3: Incentive Compatibility: Uniform Distribution, 2 bidders, 4 goods

This graph shows the minimum delta value that supports the collusion $\left(\bar{\delta}\right)$*, given the difference of the players' valuations of the goods sold in the first and last period of the game stage,* θ_2 *and* θ_1*, considering the case of the player who, according to the collusion rule, loses the good auctioned in the current period* (θ_2) *and wins only in the fourth period auction in an environment of zero reserve price and orders that cover four goods.*

5 Auctioneer Behavior

In this section, we consider how the auctioneer may act to reduce the possibility of collusion. We highlight the following instruments: increase in the number of players, schedule adjustments, changes in reserve price, and change in the tiebreaker rule.

5.1 Increase in the number of players

As we should expect, an increase in the number of bidders reduces the expected profits of cartel participants. Taking $b_0 = 0$ and $m = 2$, we showed that the expected profit of a bidder in collusion is given by:

$$\frac{1}{n}\left[2\left(1-2\left(\frac{1}{2}\right)^{n}\right)E\left(\theta F\left(\theta\right)\right)+2\left(\frac{1}{2}\right)^{n}E\left(\theta\right)\right].$$

Obviously, when $n \to \infty$, the expression above goes to zero, so that the profit from remaining in collusion goes to zero. Thus, the auctioneer would be able to extract all the winnings from the players, with or without collusion.

5.2 Schedule adjustments

We define a schedule adjustment as any change in the schedule that would limit the auctions' timeline. Such an adjustment would be an effective mechanism to avoid collusion whenever it induces a finite supergame. In this case, backward induction implies that the unique equilibrium is competition in every auction. However, in many situations this option is not available for the auctioneer: first, because many auctions are used for recurring purchases or sales – for example, in the case of local and state governments' procurement auctions; second, collusion may happen across auctions set by multiple auctioneers.

5.3 Raising the reserve price

In order to use the reserve price as a policy instrument, we modify the model by assuming that the auctioneer's valuation is private information. Hence, we define b^* as the value that the auctioneer attributes to the good to be sold, while b_0 is the reserve price. We can credibly commit to a reserve price b_0, ruling out a post-auction bargain between the winning bidder and the auctioneer. In this section, we focus on the special case in which $F(\theta)$ follows a Uniform distribution $U\,[0.1]$.

In principle, one of the ways the auctioneer may attempt to reduce the impact of collusion is by increasing the reserve price. A higher reserve price may not only reduce bidders' gain in remaining in collusion, thus increasing the incentive to deviate, but may also reduce the auctioneer's losses in case of collusion. From equation (7), considering at first the simplifying assumption that $\delta\left(\theta_i^1 - b_0\right) = \left(\theta_i^0 - b_0\right)$, we obtain the following values of $\bar{\delta}\left(b_0\right)$:

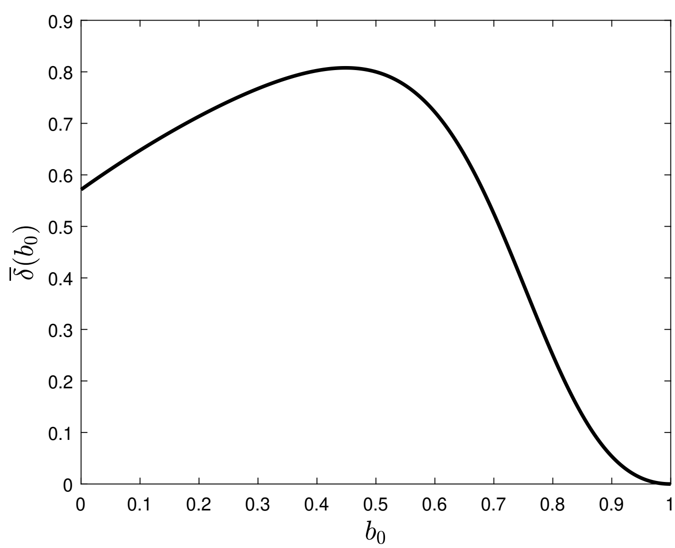

Figure 4: Relationship between reserve price and minimum required patience

Notice that the impact of b_0 on $\bar{\delta}(b_0)$ is non-linear, increasing at lower values of b_0 and decreasing after that. A potential explanation for the decline in $\bar{\delta}(b_0)$ for high values of b_0 comes from the fact that high b_0 may increase the probability that a bidder's valuation is below b_0, reducing competition and the incentive to deviate. Consequently, $\frac{\partial \overline{\delta}}{\partial b_0}$ is ambiguous. Nevertheless, the reserve price can be an instrument to maximize the expected revenue from the auctioneer. Let us now address this issue in the general case.

For completeness, in Appendix C, we show examples of the relationship between $\overline{\delta}$, θ_i^0, and θ_i^1, for the cases $b_0 = 0.4$ and $b_0 = 0.8$.

5.4 Auctioneer maximization

Let's now look at the auctioneer's maximization problem. Let's assume that the value of δ is common knowledge and that the auctioneer is committed to maintaining the agreed auction mechanism. Therefore, the auctioneer's maximization problem for the case of 2 bidders is given by:

$$\max_{b_0 \in [0,1]} \left\{ \begin{array}{c} I(\delta < \overline{\delta}(b_0)) \left\{ b^* + 2\int_{b_0}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)} - b^*\right) F(\theta) f(\theta)\, d\theta \right\} + \\ I(\delta > \overline{\delta}(b_0)) \left\{ b^* + (b_0 - b^*) \left[1 - F(b_0)^2\right] \right\} \end{array} \right\}. \tag{8}$$

where indicator functions serve to evaluate possible cases in which the collusion break is successful or not and $\overline{\delta}(b_0)$ is the minimum required value of δ for the maintenance of collusion, which, as we have already seen, depends on the reserve price.[4]

Therefore, the expression of competition-collusion revenue variation – the difference between the auctioneer's optimal profit in each case, for a given b_0 – is:

$$\Delta\Pi_L = \left\{ 2\int_{b_0}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)} - b^*\right) F(\theta) f(\theta)\, d\theta \right\} - \\ (b_0 - b^*) \left[1 - F(b_0)^2\right].$$

Proposition 6 *Considering the auctioneer's maximization problem, we can show the following results:*

a. $\Delta\Pi_L > 0$, $\forall\, b_0 \in [0, \overline{\theta}]$.

b. The auctioneer's valuation (b^*) *does not affect* $\Delta\Pi_L$.

c. Increases in b_0 *reduce* $\Delta\Pi_L$.

Therefore, Proposition 6 shows several interesting results. First of all, regardless the reserve price b_0 and the auctioneer's own valuation b^*, it is always better for the auctioneer if she can induce bidders to compete with each other. Second, even though the impact of b_0 on the incentive to collude is ambiguous, increasing b_0 does reduce the auctioneer losses due to collusion.

[4]For details on the derivation of the competition case, see Matthews (1995).

5.5 Change in tiebreaker rule

One mechanism that the auctioneer may use to try to reduce the possibility of collusion between participants is to change the asset allocation in the event of a tie, which until now was done through a draw between agents.

The possibilities for creating arbitrary rules in the event of a tie are endless and even include the possibility of creating hybrid auction mechanisms. For example, in privatization auctions in Brazil, there was a follow-up English auction between the winner of the first-price, sealed bid auction and the losers whose bids were $X\%$ less than the winning bid. In most cases, X varied between 5 and 20%.

In this section, we show that the way in which assets are allocated between cartel participants can be internalized in the collusion rule without causing major problems for the sustainability of the cartel, regardless of the rule established by the auctioneer. In this process, we maintain the collusion scheme presented, without considering the additional possibilities generated by the introduction of the tiebreaker.[5]

For simplicity, we consider the case in which there are two potential buyers and two goods to be auctioned during the stage game. Define $q = 1, 2, 3, ...$ the numbering of the current stage game (remembering that each state game has two auctions). From this, we change marginally the collusion scheme as follows:

1. *Communication Phase:* Each player sends a message about her *ordering* of preferences for the goods to be auctioned in the stage game and simultaneously receives the message from the other player, updating her beliefs regarding her opponent's ordering. Furthermore, the player who has a valuation below the reserve price for any of the goods sends a message indicating that she is not participating in the auction of this asset.

2. *Auction Phase:* Players make their moves according to the following strategy:

   (a) Consider the auction of good k: Suppose that both players have valuations above b_0. If $pos(\theta_i^k) < pos(\theta_{-i}^k)$, player i bids $b = b_0$, while her opponent bids $\bar{b} < b_0$, and vice versa. If $pos(\theta_i^k) = pos(\theta_{-i}^k)$ and we are at a stage game q and q is an odd number, player i bids $b = b_0$ while her opponent bids $\bar{b} < b_0$ if the good k is the one with the lowest order. If the good k has the greater ordering, players' roles are reversed. In the case of q even, the reasoning is symmetric, with the reversal of the players' roles (i bids on the higher order good, while $-i$ bids on the lower order good).

   (b) If any of the players break the collusion, both players revert to competition from this stage onward.

Note that the expression of the *ex ante* profit of participants in the collusion does not change, since $\Pr(q \text{ is odd}) = \Pr(q \text{ is even}) = \frac{1}{2}$, matching the case of randomization by the auctioneer.

[5] For example, the use of an English-type auction as a resource allocation instrument can generate the possibility of additional gains from collusion that we are not dealing with here.

Hence, all results we obtained about conditions for the viability of collusion are still valid. Regarding the one-shot deviation principle, the only change is that we no longer necessarily have that $\theta_i^1 > \theta_i^0$, since it is possible that $\theta_i^1 < \theta_i^0$ and even $\theta_i^0 < b_0$.[6]

Therefore, the endogenization of the allocation mechanism in the case of draws leads to an increase in the minimum δ required to support collusion in the most extreme cases, a result previously presented by Johnson and Robert (1999).[7] However, tiebreaker rules do not compromise the viability of the collusion itself in most cases.

The introduction of new players does not change the endogenization of the tiebreaker rule presented in a meaningful way. We may continue considering the period q number as a decision rule in the case of a tie in the ordering. For example, in the case of 3 players, we can consider that if $q = 1, 4, 7, ...$, player 1 must win the good with the lowest order and player 3 must win the good with the highest order. If $q = 2, 5, 8, ...$, player 2 must win the lowest order good and player 1 the highest order good. Finally, if $q = 3, 6, 9, ...$, player 3 must win the lowest ordering good and player 2 the highest order good. Note that again the profit *ex ante* does not change, while the one-shot deviation principle undergoes the same changes as in the case of 2 players.

6 Conclusion

This work presents a collusion model in repeated first-price auctions in which agents send costless messages about their preference ordering and interest in participating in the auction. In this environment, we show that the expected profit of collusion participants is greater than the profit obtained in the case of tacit static collusion. Furthermore, we show that the communication scheme presented is not strictly dominated by the dynamic collusion presented by Aoyagi (2003), showing a clear trade-off between messages' information content and the need for inefficient punishments to sustain truth-telling incentive compatibility. Regarding the asymptotic efficiency of the preference ordering communication scheme presented by Pesendorfer (2000), we show that there is a *trade-off* between efficiency and the possibility of collusion, since asymptotic efficiency demands that the patience rate approaches one asymptotically.

Regarding the auctioneer's behavior, we confirm Johnson and Robert's (1999) result about the ambiguity of the reserve price as an instrument to break collusion. The best instruments for breaking incentives for collusion are increasing the number of players and increasing the rigidity of the auctions' schedule, although these instruments may not be available to the auctioneer.

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[6]For example, in the case where there is a tie in an even period and the highest order good is in the 2nd auction.

[7]This work shows that participants in a collusion will always prefer an *ex post* tiebreaker rule to an *ex ante* rule.

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Appendix

A Calculation of bidder's expected revenue

Initially, let's assume $b_0 = 0$. The expected value for the case of the i^{th} good in a ranking of m goods is:

$$\int_0^1 \frac{m!}{(i-1)!\,(m-i)!} \theta F(\theta)^{m-i} \left[1 - F(\theta)\right]^{i-1} f(\theta)\, d\theta. \tag{A.9}$$

while the probability of winning the auction while ranking the good in the i^{th} position is:

$$\left[1 - \frac{(i-1)}{m}\right]^n - \left[1 - \left(\frac{i}{m}\right)\right]^n, \tag{A.10}$$

where n is the number of bidders in the collusive scheme.

The expected value per contract is given by:

$$\sum_{i=1}^{m} \left[\left[1 - \frac{(i-1)}{m}\right]^n - \left[1 - \left(\frac{i}{m}\right)\right]^n\right] \times \int_0^1 \frac{m!}{(i-1)!\,(m-i)!} \theta F(\theta)^{m-i} \left[1 - F(\theta)\right]^{i-1} f(\theta)\, d\theta.$$

Since the probability of each player winning the auction *ex ante* is $\frac{1}{n}$, we have that the expected return of a player per auction is:

$$\frac{1}{n} \left\{ \sum_{i=1}^{m} \left[\left[1 - \frac{(i-1)}{m}\right]^n - \left[1 - \left(\frac{i}{m}\right)\right]^n\right] \times \int_0^1 \frac{m!}{(i-1)!\,(m-i)!} \theta F(\theta)^{m-i} \left[1 - F(\theta)\right]^{i-1} f(\theta)\, d\theta \right\}.$$

For the case of two players ($n = 2$), we have:

$$\sum_{i=1}^{2} \left[\left(1 - \frac{(i-1)}{2}\right)^2 - \left(1 - \frac{i}{2}\right)^2\right] \times \int_0^1 \frac{2!}{(i-1)!\,(2-i)!} \theta F(\theta)^{i-1} \left[1 - F(\theta)\right]^{2-i} f(\theta)\, d\theta.$$

$$= \frac{3}{4} \left(\left[2E\left(\theta F(\theta)\right)\right]\right) + \frac{1}{4} \left(2\left[E(\theta) - E\left(\theta F(\theta)\right)\right]\right) = \frac{1}{2} E(\theta) + E\left(\theta F(\theta)\right).$$

As the probability of each player winning the auction *ex ante* is $\frac{1}{2}$, we have that the expected return of a player per auction is:

$$\frac{1}{2} E\left(\theta F(\theta)\right) \; + \; \frac{1}{4} E(\theta).$$

Let us now consider that $b_0 > 0$, still in the case of two players. In this case, we have:

1. Expected value of the valuation of the ordered good in 1st place:

$$\int_{b_0}^1 \frac{2!}{0!1!} \frac{\theta F(\theta) f(\theta)}{1 - F(b_0)^2} d\theta = \frac{2}{\left[1 - F(b_0)^2\right]} \int_{b_0}^1 \theta F(\theta) f(\theta)\, d\theta.$$

2. Expected value of the valuation of the ordered good in 2nd place:

$$\int_{b_0}^{1} \frac{2!}{1!0!} \frac{(\theta - b_0)\, F(\theta)^0 \left[1 - F(\theta)\right]^1 f(\theta)}{\left[1 - F(b_0)\right]^2} d\theta = 2 \int_{b_0}^{1} (\theta - b_0) \frac{\left[1 - F(\theta)\right]^1 f(\theta)}{\left[1 - F(b_0)\right]^2} d\theta.$$

However, now the probabilities regarding the ordering of the goods for the winner change, as there will be cases in which no one will want to win the auction. The probability that the good worth being purchased is ordered in 1st place becomes:

$$P\left(other\ good < b_0\right) \times 1\ + P\left(other\ good \geq b_0\right) \times P\left(good\ to\ be\ auctioned > other\ good\right) =$$
$$= F(b_0) \times 1\ +\ (1 - F(b_0)) \times \frac{1}{2}.$$

therefore, we have that the probability that the winner ordered the good in 1st place is:

$$1 -\ P\left(All\ bidders\ ranked\ the\ good\ in\ 2nd\ place\right) =\ 1 - \left(\frac{1 - F(b_0)}{2}\right)^2.$$

Thus, the expected value per contract, for each agent, is given by:

$$F(b_0)\,(1 - F(b_0)) \int_{b_0}^{1} \frac{(\theta - b_0)}{1 - F(b_0)} f(\theta)\, d\theta\ +$$
$$\frac{(1 - F(b_0))^2}{2} \left[\begin{array}{c} \left[1 - \frac{(1-F(b_0))^2}{4}\right] \frac{2}{\left[1-F(b_0)^2\right]} \int_{b_0}^{1} (\theta - b_0)\, F(\theta)\, f(\theta)\, d\theta + \\ \frac{(1-F(b_0))^2}{4} \frac{2}{\left[1-F(b_0)\right]^2} \int_{b_0}^{1} (\theta - b_0) \left[1 - F(\theta)\right]^1 f(\theta)\, d\theta \end{array} \right].$$

simplifying it, we obtain:

$$F(b_0) \int_{b_0}^{1} (\theta - b_0)\, f(\theta)\, d\theta\ + \frac{(1 - F(b_0))^2}{2} \left[\begin{array}{c} \frac{3\ + 2F(b_0)\ - F(b_0)^2}{2\left[1-F(b_0)^2\right]} \int_{b_0}^{1} (\theta - b_0)\, F(\theta)\, f(\theta)\, d\theta + \\ \frac{1}{2} \int_{b_0}^{1} (\theta - b_0) \left[1 - F(\theta)\right] f(\theta)\, d\theta \end{array} \right]$$
$$= \left[\frac{1+2F(b_0)\ + F(b_0)^{\mathbf{2}}}{4}\right] \int_{b_0}^{1} (\theta - b_0)\, f(\theta)\, d\theta\ + \frac{1 - F(b_0)}{2} \int_{b_0}^{1} (\theta - b_0)\, F(\theta)\, f(\theta)\, d\theta.$$

Considering only the incentive to collude (without taking into account the incentive to deviate), we have that the *ex ante* profit from collusion must be greater than the competition, that is:

$$\sum_{k=0}^{\infty} \delta^k \left\{ \begin{array}{c} \frac{1}{4} \left\{1\ -\ \int_0^1 F(\theta)^2 d\theta\right\} \\ + \frac{1}{4} E(\theta) \end{array} \right\}\ \geq\ \sum_{k=0}^{\infty} \delta^k \left\{1 - E\left[\theta\right] - \int_0^1 F(\theta)^2 d\theta\right\} =$$
$$\frac{1}{4} \left(1\ -\ \int_0^1 F(\theta)^2 d\theta\ + E(\theta)\right)\ \geq\ \left(1 - E\left[\theta\right] - \int_0^1 F(\theta)^2 d\theta\right).$$

which implies:

$$\int_0^1 F(\theta)^2 d\theta \ - \frac{1}{4}\int_0^1 F(\theta)^2 d\theta \ + E\left[\theta\right] \ + \frac{1}{4}E(\theta) \ \geq \ \frac{3}{4}.$$
$$E\left(\theta F\left(\theta\right)\right) \leq \frac{5}{6}E\left(\theta\right).$$

B Proofs

Proof of Proposition 1

Proof. Consider that player i has the following ranking: $pos\left(\theta_i^k\right) < pos\left(\theta_i^{k-(-1)^k}\right)$. If she sends her opponent an alternative ranking $pos\left(\theta_i^{k-(-1)^k}\right) < pos\left(\theta_i^k\right)$ she reduces the possibility of winning good k – because if player $-i$ orders $pos\left(\theta_{-i}^k\right) < pos\left(\theta_{-i}^{k-(-1)^k}\right)$, according to the collusion rules, player $-i$ must win the auction for good k, which increases the probability of gaining good $k-(-1)^k$, which is not so desiable. Given the symmetry in *ex ante* terms of the players, these variations must have the same probability, generating a loss in terms of expected return. ■

Proof of Proposition 2

Proof. Before we start, let's show an auxiliary result that will help us with the proof.

Lemma A.1 *If $F_1(\theta)$ second-order stochastically dominates $F_2(\theta)$, we have the following expression:*

$$\int_0^{\overline{\theta}} F_1\left(\theta\right)^2 d\theta \geq \int_0^{\overline{\theta}} F_2\left(\theta\right)^2 d\theta. \tag{A.11}$$

Proof. From second-order stochastical dominance, we have:

$$\int_0^{\overline{\theta}} \left(F_1\left(x\right) - F_2\left(x\right)\right) dx = 0. \tag{A.12}$$

and

$$\int_0^{\theta} \left(F_1\left(x\right) - F_2\left(x\right)\right) dx \leq 0, \ \forall\theta. \tag{A.13}$$

Notice that:

$$\int_0^{\overline{\theta}} F_1\left(\theta\right)^2 d\theta - \int_0^{\overline{\theta}} F_2\left(\theta\right)^2 d\theta = \int_0^{\overline{\theta}} \left(F_1\left(\theta\right) + F_2\left(\theta\right)\right)\left(F_1\left(\theta\right) - F_2\left(\theta\right)\right) d\theta$$

from integration by parts, we have:

$$\begin{gathered}\int_0^{\overline{\theta}} \left(F_1\left(\theta\right) + F_2\left(\theta\right)\right)\left(F_1\left(\theta\right) - F_2\left(\theta\right)\right) d\theta = \\ = \left\{ \begin{array}{c} \left(F_1\left(\theta\right) + F_2\left(\theta\right)\right)\int_0^{\theta} \left(F_1\left(x\right) - F_2\left(x\right)\right) dx|_0^{\overline{\theta}} \\ -\int_0^{\overline{\theta}} \left[\left(f_1\left(\theta\right) + f_2\left(\theta\right)\right)\int_0^{\theta} \left(F_1\left(x\right) - F_2\left(x\right)\right) dx\right] d\theta \end{array} \right\}\end{gathered} \tag{A.14}$$

given property (A.12), equation (A.14) simplifies to:

$$\int_0^{\overline{\theta}} F_1(\theta)^2 \, d\theta - \int_0^{\overline{\theta}} F_2(\theta)^2 \, d\theta = -\int_0^{\overline{\theta}} \left[\left(f_1(\theta) + f_2(\theta) \right) \int_0^{\theta} \left(F_1(x) - F_2(x) \right) dx \right] d\theta \tag{A.15}$$

since $f_i(\theta)$ is a probability density function, we have that $f_i(\theta) \geq 0, \forall i \in \{1, 2\}$. Moreover, from property (A.13), we have that $\int_0^{\theta} \left(F_1(x) - F_2(x) \right) dx \leq 0, \forall \theta$. Consequently, we must have:

$$\int_0^{\overline{\theta}} F_1(\theta)^2 \, d\theta - \int_0^{\overline{\theta}} F_2(\theta)^2 \, d\theta \geq 0$$

concluding our proof. ■

Then, from equation (6), we have that:

$$\bar{\delta}_i = \frac{4\left(1 - E[\theta] - \int_0^1 F_i(\theta)^2 d\theta\right)}{\left(1 - \int_0^1 F_i(\theta)^2 d\theta + E(\theta)\right)}, \ i \in \{1, 2\}$$

where we are already taking into account that $E(\theta)$ is the same for distributions F_1 and F_2 due to second order stochastic dominance. Then, we have that:

$$\bar{\delta}_1 - \bar{\delta}_2 = -\frac{8E(\theta)\left[\int_0^1 F_1(\theta)^2 \, d\theta - \int_0^1 F_2(\theta)^2 \, d\theta\right]}{\left(1 - \int_0^1 F_1(\theta)^2 d\theta + E(\theta)\right)\left(1 - \int_0^1 F_2(\theta)^2 d\theta + E(\theta)\right)} \leq 0$$

where the inequality comes from the result presented in Lemma A.1. ■

Lemma A.2 *Consider two distribution functions F_1 and F_2, in an environment in which collusion is* ex ante *profitable. Even if $\delta(\theta_i^1) \neq \theta_i^0$, we have that $\overline{\delta}_1 \leq \overline{\delta}_2$, where δ_i is the discount rate relative to F_i.*

Proof. From equation (4), for distribution $F_i(\cdot)$, $i \in \{1, 2\}$, we have that collusion is sustainable at discount rate δ if:

$$\begin{aligned} &\delta(\theta_i^1) + \tfrac{\delta^2}{1-\delta}\left[\tfrac{1}{4}\left(1 + E[\theta] - \textstyle\int_0^1 F_i(\theta)^2 d\theta\right)\right] \\ &\geq (\theta_i^0 - \varepsilon) + \tfrac{\delta}{1-\delta}\left(1 - E[\theta] - \textstyle\int_0^1 F_i(\theta)^2 d\theta\right) \end{aligned}$$

Now, consider the case of $\bar{\delta}_2$. Then, we have:

$$\begin{aligned} &\bar{\delta}_2(\theta_i^1) + \tfrac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left[\tfrac{1}{4}\left(1 + E[\theta] - \textstyle\int_0^1 F_2(\theta)^2 d\theta\right)\right] \\ &= (\theta_i^0 - \varepsilon) + \tfrac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(1 - E[\theta] - \textstyle\int_0^1 F_2(\theta)^2 d\theta\right) \end{aligned}$$

manipulating it, we have:

$$\begin{aligned}&\bar{\delta}_2(\theta_i^1)+\tfrac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left[\tfrac{1}{4}\left(1+E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta+\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)\right]\\&=(\theta_i^0-\varepsilon)+\tfrac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(1-E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta+\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)\end{aligned}$$

rearranging it:

$$\begin{aligned}&\bar{\delta}_2(\theta_i^1)+\tfrac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left[\tfrac{1}{4}\left(1+E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta\right)\right]+\tfrac{1}{4}\tfrac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left(\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)\\&=(\theta_i^0-\varepsilon)+\tfrac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(1-E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta\right)+\tfrac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)\end{aligned}$$

since $+\frac{1}{4}\frac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left(\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)\leq\frac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(\int_0^1 F_1(\theta)^2d\theta-\int_0^1 F_2(\theta)^2d\theta\right)$, we must have:

$$\begin{aligned}&\bar{\delta}_2(\theta_i^1)+\tfrac{\bar{\delta}_2^2}{1-\bar{\delta}_2}\left[\tfrac{1}{4}\left(1+E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta\right)\right]\\&\geq(\theta_i^0-\varepsilon)+\tfrac{\bar{\delta}_2}{1-\bar{\delta}_2}\left(1-E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta\right)\end{aligned}\tag{A.16}$$

Consequently, at $\bar{\delta}_2$, collusion is strictly preferred under $F_1(\cdot)$. To conclude the proof, we just need to show that once the restriction is satisfied for a given δ, it must be satisfied for every $\delta'>\delta$. To simplify the notation, define:

$$\pi_{\text{Collusion}}=\frac{1}{4}\left(1+E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta\right)$$

and

$$\pi_{\text{Competition}}=1-E\left[\theta\right]-\int_0^1 F_1(\theta)^2d\theta$$

given that collusion is *ex ante* profitable, we must have $\pi_{\text{Collusion}}>\pi_{\text{Competition}}$. Then, the one-shot deviation principle can be rewritten as:

$$\delta(\theta_i^1)+\tfrac{\delta^2}{1-\delta}\pi_{\text{Collusion}}\geq(\theta_i^0-\varepsilon)+\tfrac{\delta}{1-\delta}\pi_{\text{Competition}}\tag{A.17}$$

Notice that:

$$\frac{\delta^2}{1-\delta}=\frac{\delta^2-\delta+\delta}{1-\delta}=\frac{\delta-\delta(1-\delta)}{1-\delta}=\frac{\delta}{1-\delta}-\delta$$

so we can rewrite equation (A.17) as:

$$\delta\left(\theta_i^1-\pi_{\text{Collusion}}\right)+\frac{\delta}{1-\delta}\left[\pi_{\text{Collusion}}-\pi_{\text{Competition}}\right]\geq\theta_i^0-\varepsilon\tag{A.18}$$

therefore, if $\tilde{\delta}$ satisfies the one-shot deviation principle, we must have that $\forall\delta\geq\tilde{\delta}$:

$$\theta_i^1-\pi_{\text{Collusion}}+\frac{1}{1-\delta}\left[\pi_{\text{Collusion}}-\pi_{\text{Competition}}\right]>0\tag{A.19}$$

Finally, define the right-hand side of (A.18) as Φ. Then, we have that:

$$\frac{\partial \Phi}{\partial \delta} = \theta_i^1 - \pi_{\text{Collusion}} + \frac{1}{(1-\delta)^2}\left[\pi_{\text{Collusion}} - \pi_{\text{Competition}}\right]$$

therefore, if $\delta > \tilde{\delta}$, we must have that $\frac{\partial \Phi}{\partial \delta} > 0$ and collusion is strictly preferred.

Finally, going back to expression (A.16), based on the above results, we must conclude that $\bar{\delta}_1 < \bar{\delta}_2$. ■

Proof of Proposition 3

Proof. The mechanism with identical bids presented by McAfee and McMillan (1992) induces the following expected profits:

$$\begin{aligned} M\&M &= \frac{1+F(b_0)}{2}\int_{b_0}^{1}(1-F(\theta))\,d\theta \\ &= \left[\frac{1+F(b_0)}{2}\right]\left[(1-b_0) - \int_{b_0}^{1} F(\theta)\,d\theta\right]. \end{aligned} \tag{A.20}$$

In contrast, once we introduce the communication mechanism through ordering preferences, the expected profit becomes:

$$\begin{aligned} &\left[\frac{1+F(b_0)}{2}\right]^2\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)\,d\theta\right\} + \frac{1-F(b_0)}{4}\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)^2\,d\theta\right\} \\ &\quad = \left[\frac{1+F(b_0)}{2}\right] M\&M + \frac{1-F(b_0)}{4}\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)^2\,d\theta\right\}. \end{aligned} \tag{A.21}$$

Then the difference between (A.21) and (A.20) is:

$$\Delta(b_0) = \left\{ \begin{array}{c} \frac{1-F(b_0)}{4}\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)^2\,d\theta\right\} \\ -\left[\frac{1-F(b_0)^2}{4}\right]\left[(1-b_0) - \int_{b_0}^{1} F(\theta)\,d\theta\right] \end{array} \right\}. \tag{A.22}$$

We must show that $\Delta(b_0) \geq 0$, $\forall b_0$. First of all, note that $\Delta(0) > 0$ and $\Delta(1) = 0$.

Taking the derivative of (A.22), we have:

$$\begin{aligned} \frac{\partial \Delta}{\partial b_0} &= -\frac{f(b_0)}{4}\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)^2\,d\theta\right\} + \frac{1-F(b_0)}{4}\left\{-1 + F(b_0)^2\right\} \\ &\quad + \frac{F(b_0)\,f(b_0)}{2}\left\{(1-b_0) - \int_{b_0}^{1} F(\theta)\,d\theta\right\} + \left[\frac{1-F(b_0)^2}{4}\right]\left\{1 - F(b_0)\right\}. \end{aligned}$$

Simplifying it, we have:

$$\frac{\partial \Delta}{\partial b_0} = \frac{f(b_0)}{4}\left\{\int_{b_0}^{1}[1-F(\theta)]\left\{\underbrace{2F(b_0)-[1+F(\theta)]}_{<0}\right\}d\theta\right\} < 0.$$

Therefore, $\Delta(b_0)$ is non-negative and strictly decreasing. ■

Proof of Proposition 4

Proof. The lower bound in the bidder's *ex ante* expected payoff in Aoyagi (2003)[8] is given by:

$$u^d > L \equiv \frac{E(\theta)}{E(\theta)+2E(\theta F(\theta))}E(\theta F\theta) + \frac{2E(\theta F(\theta))}{E(\theta)+2E(\theta F(\theta))}\frac{E(\theta)}{2}.$$

in contrast, in our model with an ordering communication system, for the case of two bidders and two goods per period – $m = n = 2$ – we have:

$$u^s = \frac{1}{2}E(\theta F(\theta)) + \frac{1}{2}\frac{E(\theta)}{2}. \tag{A.23}$$

Given (A.23), we have that $u^s > L$. Hence, we are unable to rank u^d and u^s. ■

Proof of Proposition 5

Proof. While considering the one-shot deviation principle, let's consider the case in which a bidder's ranking is a shift in an opponent's ranking. For example, consider the case in which a bidder's ranking is $(\theta_1, \theta_2, \theta_3, \theta_4, ..., \theta_m)$, while an opponent's ranking is given by $(\theta_2, \theta_3, \theta_4, ..., \theta_m, \theta_1)$, where θ_1 is the good auction in the m^{th} period, while θ_2 is sold in the 1^{st} period of the stage game. Consequently, the bidder with ranking $(\theta_1, \theta_2, \theta_3, \theta_4, ..., \theta_m)$ does not deviate from the collusion strategy if:

$$\delta^{m-1}(\theta_1) + \frac{1}{2}\frac{\delta^m}{1-\delta}\left\{\begin{array}{c} \sum_{i=1}^{m}\left[\begin{array}{c}\left[1-\frac{(i-1)}{m}\right]^2 \\ -\left[1-\left(\frac{i}{m}\right)\right]^2\end{array}\right]\times \\ \int_0^1 \frac{m!}{(i-1)!(m-i)!}\theta F(\theta)^{m-i}[1-F(\theta)]^{i-1}f(\theta)\,d\theta \end{array}\right\}$$
$$\geq \theta_2 + \frac{\delta}{1-\delta}\left[\int_0^1 F(\theta)d\theta - \int_0^1 F(\theta)^2 d\theta\right].$$

Notice that, as $m \to \infty$, the right-hand side of the above expression converges to $\theta_2 + g^*$. In contrast, as $m \to \infty$, $\delta^m \to 0$, so the left-hand side converges to 0 and the inequality cannot be satisfied. Hence, even though $m \to \infty$ makes payoffs asymptotically efficient, it also makes collusion unsustainable. ■

Proof of Lemma 1

[8] See Aoyagi (2003), page 13.

Proof. Similar to the proof of Lemma A.2, define $\pi_{\text{Collusion}}(b_0)$ and $\pi_{\text{Competition}}(b_0)$ as:

$$\pi_{\text{Collusion}}(b_0) = \frac{7 - 10b_0 + 2b_0^3 + b_0^4}{24}$$

and

$$\pi_{\text{Competition}}(b_0) = \frac{2b_0^3 - 3b_0^2 + 1}{6}$$

given that collusion is *ex ante* profitable, we must have $\pi_{\text{Collusion}}(b_0) > \pi_{\text{Competition}}(b_0)$. Then, the one-shot deviation principle can be rewritten as:

$$\delta\left(\theta_i^1 - b_0 - \pi_{\text{Collusion}}\right) + \frac{\delta}{1-\delta}\left[\pi_{\text{Collusion}}(b_0) - \pi_{\text{Competition}}(b_0)\right] - \theta_i^0 + b_0 + \varepsilon \geq 0$$

define the LHS of the above inequality as Φ, as we showed in the proof of Lemma A.2, for any δ that satisfies the one-shot deviation principle,[9] we must have $\frac{\partial \Phi}{\partial \delta} > 0$.

Finally, notice that:

$$\frac{\partial \Phi}{\partial \theta_i^1} = \delta > 0$$

and

$$\frac{\partial \Phi}{\partial \theta_i^0} = -1 < 0$$

then, from the implicit function theorem, we have that:

$$\frac{\partial \overline{\delta}}{\partial \theta_i^0} = -\frac{\frac{\partial \Phi}{\partial \theta_i^0}}{\frac{\partial \Phi}{\partial \delta}} > 0$$

and

$$\frac{\partial \overline{\delta}}{\partial \theta_i^1} = -\frac{\frac{\partial \Phi}{\partial \theta_i^1}}{\frac{\partial \Phi}{\partial \delta}} < 0$$

concluding the proof. ■

Proof of Proposition 6

Proof. Let's start with item *a*. Simplifying the expression for $\Delta\Pi_L$, we have:

$$\Delta\Pi_L(b_0) = \left\{2\int_{b_0}^{\overline{\theta}}\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)F(\theta)\,f(\theta)\,d\theta\right\} - b_0\left[1 - F(b_0)^2\right] \tag{A.24}$$

from integration by parts, the expression simplifies to:

$$\Delta\Pi_L(b_0) = 2\int_{b_0}^{\overline{\theta}}\theta f(\theta)\left(1 - F(\theta)\right)d\theta - b_0\left[1 - F(b_0)\right]^2. \tag{A.25}$$

[9] In fact, this is true even within a neighborhood of $\overline{\delta}$, since $\theta_i^0 - b_0 - \varepsilon > 0$.

Furthermore, note that $\Delta\Pi_L(0) = 2[E(\theta) - E(\theta F(\theta))] > 0$ *e* $\Delta\Pi_L(1) = 0$. Moreover, taking the first and second derivatives of $\Delta\Pi_L$ with respect to b_0, we obtain:

$$\frac{d\Delta\Pi_L(b_0)}{db_0} = -[1 - F(b_0)]^2 < 0, \forall\, b_0 \in (0,1) \tag{A.26}$$

and

$$\frac{d^2\Delta\Pi_L(b_0)}{d(b_0)^2} = 2f(b_0)(1 - F(b_0)) > 0, \forall\, b_0 \in (0,1) \tag{A.27}$$

Consequently, $\Delta\Pi_L(b_0)$ is strictly decreasing and convex in $(0,1)$. Given the values of $\Delta\Pi_L$ for $b_0 = 0$ and $b_0 = 1$, we have that $\Delta\Pi_L > 0$, $\forall b_0 \in (0,1)$, concluding the proof of part *a*.

Now let's consider item *b*. From the simplification of the expression for $\Delta\Pi_L$ presented in equation (A.24), we can already see that b^* cancels out.

Finally, in order to pin down item *c*, notice that equation (A.26) shows that $\frac{d\Delta\Pi_L(b_0)}{db_0} < 0$, concluding the proof. ■

C Examples: Uniform distribution case with positive reserve prices

We start by considering the case $b_0 = 0.4$ and assume that θ_i^0 and θ_i^1 are larger than b_0. Figures 5 and 6 show the results.

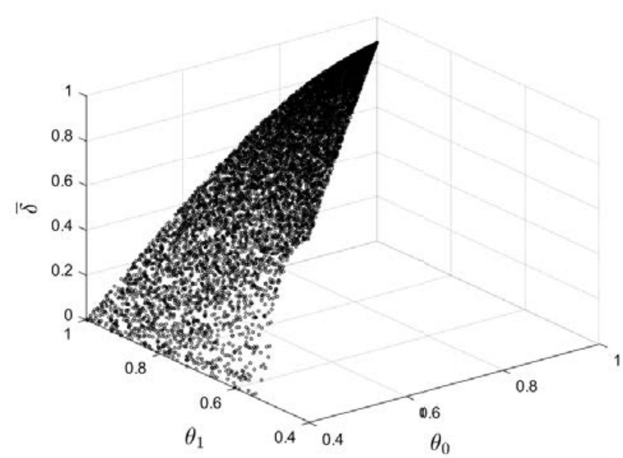

Figure 5: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods – $b_0 = 0.4$

This graph shows the minimum delta value that supports the collusion $(\bar{\delta})$, given the auction participants' valuations for the goods that will be sold in the stage game, θ_0 and θ_1, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current stage (θ_0) and wins the auction of the following stage, when reserve price is equal to 0.4 *and orders that include only two goods.*

Considering $\Delta\theta = \theta_i^1 - \theta_i^0$ and $b_0 = 0.4$, we have:

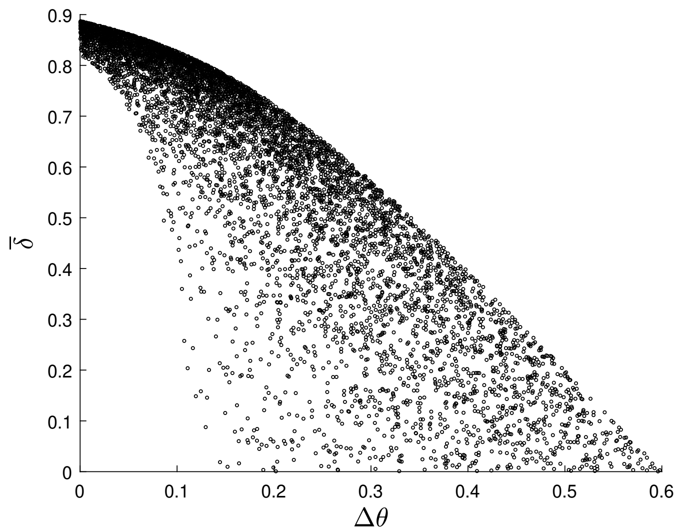

Figure 6: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods – $b_0 = 0.4$

This graph shows the minimum delta value that supports the collusion $(\bar{\delta})$*, given the difference in the players' valuations of the good sold in the stage game,* θ_0 *and* θ_1*, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current stage* (θ_0) *and wins the auction of the following stage, when the reserve price is equal to* 0.4 *and orders that cover only two goods.*

Similarly, in the case in which the reserve price is $b_0 = 0.8$, we have:

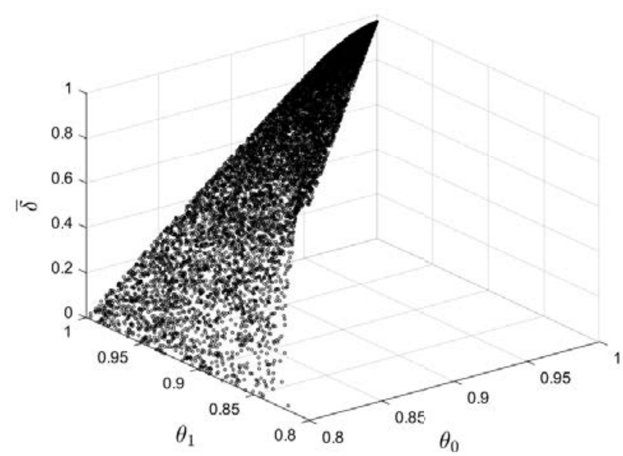

Figure 7: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods – $b_0 = 0.8$

This graph shows the minimum delta value that supports the collusion $(\bar{\delta})$*, given the auction participants' valuations for the goods that will be sold in the stage game,* θ_0 *and* θ_1*, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current stage* (θ_0) *and wins the auction of the following stage, when the reserve price is equal to* 0.8 *and orders that include only two goods.*

Considering $\Delta\theta = \theta_i^1 - \theta_i^0$ and $b_0 = 0.8$, we have:

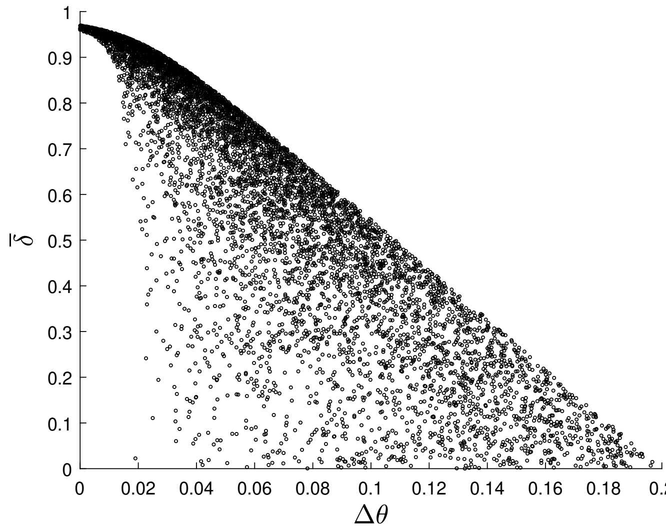

Figure 8: Incentive Compatibility: Uniform Distribution, 2 bidders, 2 goods – $b_0 = 0.8$

This graph shows the minimum delta value that supports the collusion $\left(\bar{\delta}\right)$, given the difference in the player's valuations of the good sold in the stage game, θ_0 and θ_1, considering the case of the player who, due to the collusion rule, loses the good auctioned in the current stage (θ_0) and wins the auction of the following stage, when the reserve price is equal to 0.8 *and orders that cover only two goods.*